The Combined Continuous Time Random Walk in Position and Momentum Space as Model for Anomalous Transport

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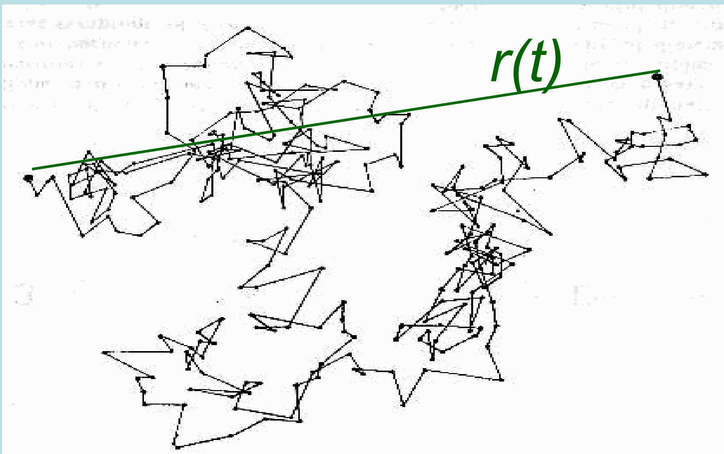
Anomalous diffusion, defined

- Anomalous diffusion is usually characterized by the scaling of the mean-square displacement (MSD)

$$\langle r^2(t) \rangle \propto t^\gamma, \quad \gamma \neq 1$$

with r the distance from the initial position, which describes how far, on the average, particles move in time. The following cases are discerned,

$\gamma > 1$: super-diffusion ($\gamma=2$: ballistic diffusion)
$\gamma < 1$: sub-diffusion
$\gamma = 1$: classical diffusion



Anomalous diffusion, observed

- Anomalous diffusion in confined plasmas is seldom directly observed as $\langle r^2(t) \rangle \propto t^\gamma, \quad \gamma \neq 1$ but in the form of strange, unexpected, anomalous plasma behavior, such as:
- Profile consistency: the density or temperature profile stays close to a certain shape, also if the plasma is distorted, e.g. by off axis heating, or fuel-injection.
- The reaction of the plasma to off-axis fueling:
  (i) Despite of off-axis injection of particles, the profiles (density etc.) are still peaked at the center.
- The reaction of the plasma to off-axis heating/cooling:
  (i) Cooling at the edge can lead to a temperature rise at the center [e.g. Tamura et al., PoP **12**, 110705 (2005)].
  (ii) During heating at edge, there can be an inwards electron heat flux, in direction of increasing temperature ('uphill').
  (iii) During dominating off-axis heating, the electron temperature remains still peaked at the center.
  [e.g. Luce et al., PRL **68**, 52 (1992)]
- Rapid (ballistic) transport phenomena:
  (i) Sudden cooling at edge may lead to a temperature drop that propagates inwards with constant velocity ($r=vt$, $r^2 = v^2t^2$, $\langle r^2() \rangle \sim t^2$, super-diffusion).
  (ii) Heat pulses generated at the center may propagate outwards with constant velocity.

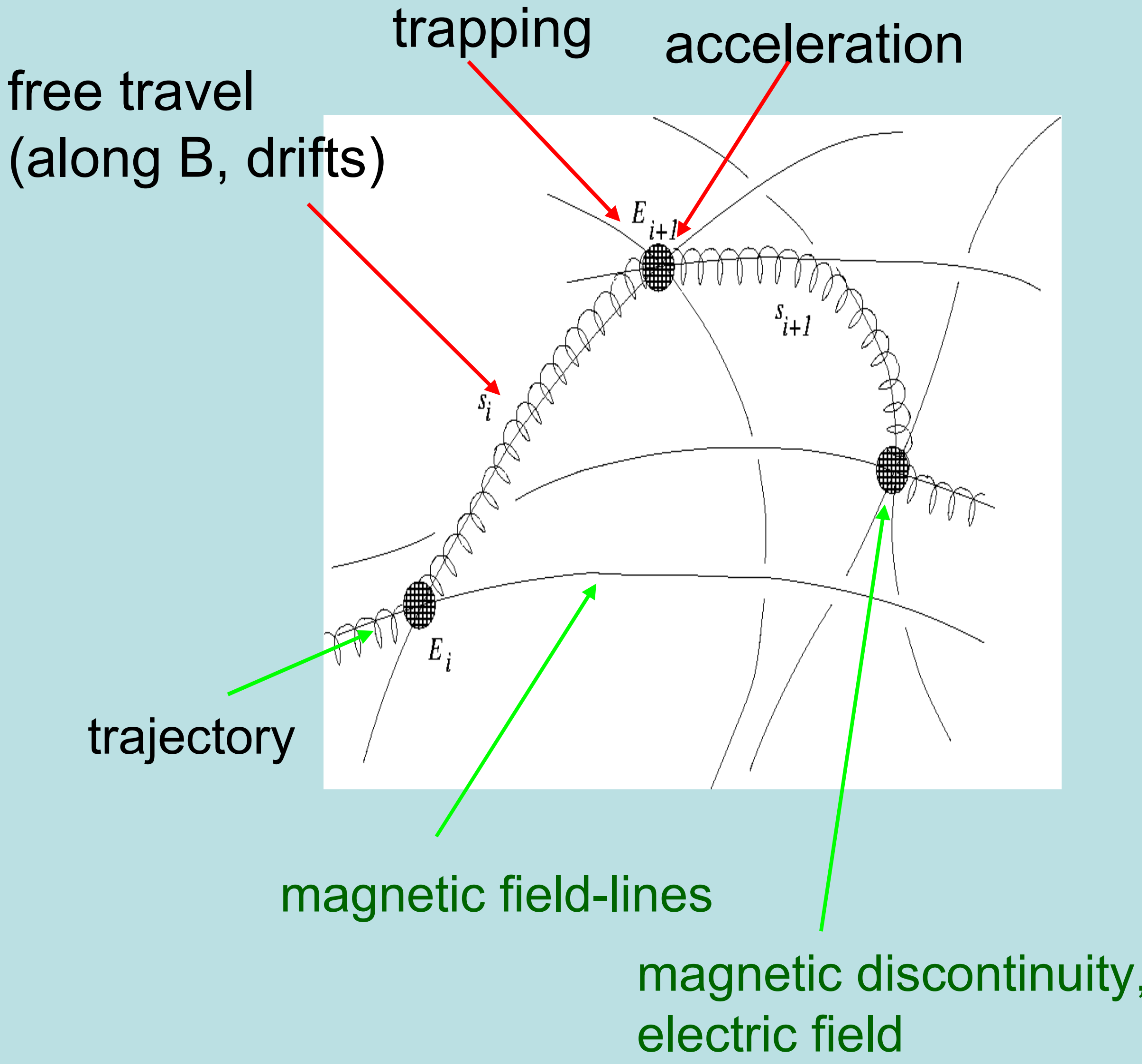
Continuous Time Random Walk (CTRW)

- It has been shown that Continuous Time Random Walk (CTRW; e.g. [1], [2]) can successfully model observed phenomena of anomalous transport in confined plasmas, such as uphill transport in response to off-axis fueling, or rapid phenomena of energy transport (e.g. [3], [4]).
- A main characteristic of the CTRW is that it is a non-local approach to diffusion in both space and time (non-Markovian), which allows it to model a wide range of sub- as well as super-diffusive phenomena.
- CTRW is closely related to the fractional Fokker Planck or diffusion equation (which, in short: is a simple diffusion equation with derivatives replaced by fractional derivatives, and where fractional derivatives are non-local operators).
- The classical random walk (RW) problem was treated in 1900 by Bachelier (stock markets), and in 1905 by Einstein (Brownian motion). They developed an integral equation, the Einstein Bachelier equation, that describes the RW problem, and which under certain assumptions can be solved analytically, yielding in general normal diffusion. The CTRW was introduced by¨[1] (waiting or trapping model), and in a modified version by Shlesinger & Klafter (1989; velocity model).

[1] Montroll, E.W., Weiss, G.H., J. Math. Phys. **6**, 167 (1965)
[2] Balescu, R., Phys. Rev. E **51**, 4807 (1995)
[3] van Milligen, B.Ph., Sanchez, R., Carreras, B.A., PoP **11**, 2272 (2004)
[4] van Milligen, B.Ph., Carreras, B.A., Sanchez, R., PoP **11**, 3787 (2004)

Random walk of plasma particles

In a turbulent plasma, a particle can freely travel along the magnetic field or drift across the magnetic field lines, it can locally be trapped, or it can locally be accelerated at magnetic discontinuities (electric fields).



The Combined CTRW in position and momentum space C-CTRW

- So-far, all applications of CTRW were done in position space, the evolution of the velocity was not included, i.e. there was no information on velocity space, the energetics etc.
- The reason was to keep the formalism simpler, also in view of the mostly analytical work done on asymptotic solutions.
- Our aim is to extend the CTRW to include, besides position space, also momentum space (combined CTRW: C-CTRW), which will be a more realistic, dynamic model.
- The velocities of particles change in collisions and in interactions with localized electric fields, so that the inclusion of momentum space leads to a more realistic random walk in position space itself.
  The C-CTRW is able to model phenomena like heating, cooling, acceleration of particle populations, the evolution of the temperature etc.

The C-CTRW in position and momentum space: definition

- Position and momentum are considered.
- The position $x_{n,}$ of a particle after n steps (at time t_n) is

□

$$x_n = \Delta x_n + \Delta x_{n-1} + \Delta x_{n-2} + \ \dots \ + \Delta x_1 + x_0$$

where Δx_i is the random jump increment

- The time t_n after n steps is

$$t_n = \Delta t_n + \Delta t_{n-1} + \Delta t_{n-2} + \ \dots \ + \Delta t_1$$

where Δt_i is the random temporal increment

- Newly in the C-CTRW, the momentum p_n of a particle after n steps is

$$p_n = \Delta p_n + \Delta p_{n-1} + \Delta p_{n-2} + \ \dots \ + \Delta p_1 + p_0$$

where Δp_i is the random change (increment) in momentum.

- To specify the random walk completely, we still need to give the distribution of jump increments Δx, momentum increments Δp, and of temporal increments Δt, $q_3(\Delta x, \Delta p, \Delta t)$, i.e. the probability to make a jump Δx and to spend a time Δt in the jump and to change the momentum by Δp
- The solution to be determined is in the form of the probability distribution $P(x,p,t)$ that a particle is at position x and has momentum p at time t.

The distribution of increments

- The distribution of increments $q_3(\Delta x, \Delta p, \Delta t)$ is here specified to the simplest possible, still meaningful form, following the 'velocity model' of the standard CTRW,

  $$q_3(\Delta x,\Delta p,\Delta t) = \delta(\Delta t - \Delta x/v)\, q_x(\Delta x)\, q_p(\Delta p)$$

  Here then, Δx and Δp are independent, and Δt is given through Δx and v, *where* v is a function of the instantaneous momentum p and is variable (i.e. a random variable; e.g. in the non-relativistic case $v = p/m$).
  Δt is thus the time a particle spends in a free flight,

  $$\Delta t = \Delta x/v.$$

  [Even simpler would be the 'waiting/trapping model',

  $$q_3(\Delta x, \Delta p, \Delta t) = q_t(\Delta t)\, q_x(\Delta x)\, q_p(\Delta p),$$

  but the momentum space and position space do not communicate in this case, they could be treated separately, by two CTRW equations]

The C-CTRW equations

- The C-CTRW equations are derived from the CTRW equations by adding – carefully – the momentum. The final equations of the C-CTRW are as follows:
- The probability *P(x,p,t)* to be at *x* with momentum *p* at time *t* is given as

$$P(x,p,t) = \int d\Delta p \int d\Delta x \int_0^t d\Delta t\, Q(x-\Delta x, p-\Delta p, t-\Delta t)\, \Phi(\Delta x, \Delta t, v)\, q_p(\Delta p)$$

- where the auxiliary function *Q* is the distribution of turning points, and it is determined as

$$\begin{aligned} Q(x,p,t) \;=\; & \int d\Delta p \int d\Delta x \int_0^t dt\, Q(x-\Delta x, p-\Delta p, t-\Delta t) \\ & \times \delta(\Delta t - \Delta x/v)\, q_x(\Delta x)\, q_p(\Delta p) + \delta(t) P(x,p,0) + S(x,p,t) \end{aligned}$$

  with *S(x,p,t)* a source term and *P(x,p,0)* the initial condition.
- Finally, *Φ* is the probability to make a jump of length *Δx* or longer and of duration *Δt* or longer, and to be at time *Δt* at position *Δx,*

$$\Phi(\Delta x, \Delta t, v) = \frac{1}{2}\,\delta(|\Delta x| - v\Delta t) \int_{|\Delta x|}^{\infty} dx' \int_{\Delta t}^{\infty} dt'\, \delta(t' - |x|'/v)\, q_x(x')$$

  Note that *Φ* is given, it is derived from the known $q_x(\Delta x)$.
- The equations are implicit integral equations, of the convolution type, they are non-local in space and non-local in time, i.e. non-Markovian, and they allow non-Gaussian, anomalous diffusion, depending on the distributions of increments

How to solve the C-CTRW equations I

- For integral equations of a convolution type, a standard way to solve them is with Fourier transform in space and Laplace transform in time. The velocity model though has *t*-dependent integration limits for Δx

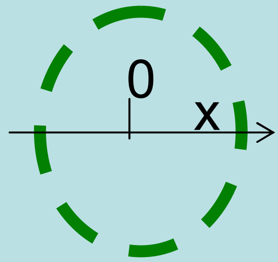
$$
\begin{aligned}
Q(x,p,t) \;=\; & \int d\Delta p \int_{max[\,|v|t\,,\,-L]}^{min[\,|v|t\,,\,L]} d\Delta x \int_0^t dt\, Q(x-\Delta x, p-\Delta p, t-\Delta t) \\
& \times \delta(\Delta t - \Delta x/v)\, q_x(\Delta x)\, q_p(\Delta p) + \delta(t) P(x,p,0) + S(x,p,t)
\end{aligned}
$$

  [at time *t*, the time Δt spent in a jump cannot be larger than t: $\Delta t \le t \rightarrow v\Delta t \le vt \rightarrow v\Delta t = \Delta x \rightarrow \Delta x \le vt$]
  There are no convolutions in Δx and Δt anymore, so that the Fourier Laplace transforms are not applicable, as in the simple CTRW waiting model. Analytic (asymptotic) solutions are therefore difficult.
- Often, the integral equations are manipulated to become only one integral equation, directly for *P*, or an integro-differential equation for *P* (generalized master equation), through reformulating in Laplace space:
  This is not possible here, again since Fourier and Laplace-transforms are not applicable.

How to solve the C-CTRW equations II

- Even if Fourier Laplace (FL) transforms were applicable, the FL transforms can yield only asymptotic solution for $<r^2(t)>$:
  (i) It is though preferable to have more information, i.e. to have the complete distribution $P(x,p,t)$ available.
  (ii) Monte-Carlo simulations show that the transient state can be important and it can be drastically different from the asymptotic, large x, large p, large t state.
- We are thus forced to solve the equations numerically. The equations are integral equations of a Volterra type, second kind, with unknown P and Q, where Q plays an auxiliary role, and where the difficult equation is the one for Q that determines Q implicitly.
- We apply a pseudospectral method, where the unknown functions are expanded in terms of Chebyshev polynomials, which allows to achieve high precision with reasonable computing time and memory use.
- (The Nystrom method would need a large amount of computing time and memory to achieve reasonable precision.)
- Monte-Carlo simulations are done to check results derived with other methods.

Application



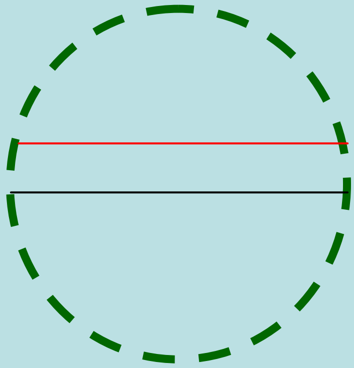
- We consider diffusion along the minor radius, we thus choose the x coordinate along the minor radius, and the system is finite in the x-direction, -2 m $\leq x \leq 2$ m ($\approx$ ITER)
- p is the momentum along x, and the p-range is large (to contain most of the initial Maxwellian).
- The particles are considered to be electrons, with temperature 8 keV ($\approx$ at ITER).
- The distribution of the momentum increments $q_p(\Delta p)$ is a Gaussian throughout the following, i.e. we allow only small increments (classical diffusion) in p.
- The Δp are assumed to be caused by an electric field E_x, $\Delta p = E_x \langle t_{acc} \rangle |e|$, [from $dp/dt = eE_x$, with E_x const], with $\langle t_{acc} \rangle$ the mean acceleration time and $|e|$ the electron charge, so that we have for the standard deviation $\sigma_{\Delta p} = \sigma_E \langle t_{acc} \rangle |e|$,
  and we here choose $\sigma_E = 2\ 10^{-1}$ V/m ($\approx 10^3 \times$ Dreicer field for $n_0 = 10^{14}$ cm^{-3})
- For the spatial increments Δx we consider two cases for the distribution $q_x(\Delta x)$
  - either a Gaussian distribution of the increments, i.e. only small increments are allowed ($\sigma_{\Delta x} = 0.2$ m), in order to model classical diffusion,
  - or $q_x(\Delta x)$ has a power-law tail, here with index 1.2,
    $$q_x(\Delta x) \sim \Delta x^{-1.2} \qquad \text{(for } \Delta x \geq 0.01\text{m)}$$
    i.e. also large increments are allowed and possible.
- The initial distribution is $P(x,p,t=0) = 0$.

Comparison with Monte Carlo simulations

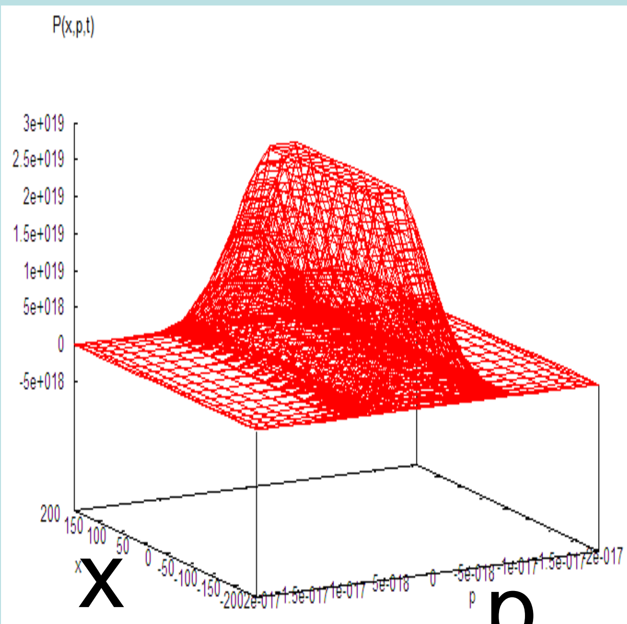
$$S_B(x,p,t) = S_0 \frac{1}{\sqrt{2\pi m_e k_B T}} e^{-\frac{p^2}{2 m_e k_B T}}$$

- We consider constant loading with a uniform constant background source, which is a Maxwellian in the *p*-direction [T[keV] = 8 keV, $S_0 = 10^{-5}$], and for the spatial increments we assume $q_x(\Delta x) \sim \Delta x^{-1.2}$.
- The solution of the C-CTRW equation is *P(x,p,t),* from which we can determine the density distribution,

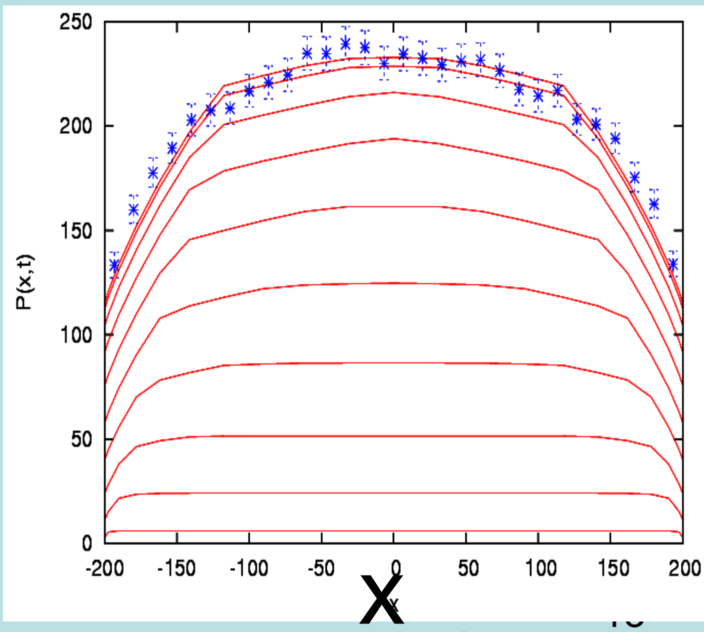
  $n(x,t) = \int dp\, P(x,p,t).$



The solution *P(x,p,t),* where each surface corresponds to a different time



The density *n(x,t),* for different times



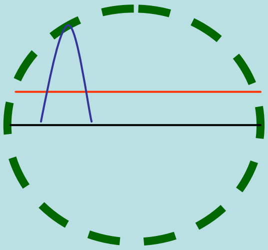
red: num. sol.
blue: MC sim.

Off-axis fueling with cold plasma

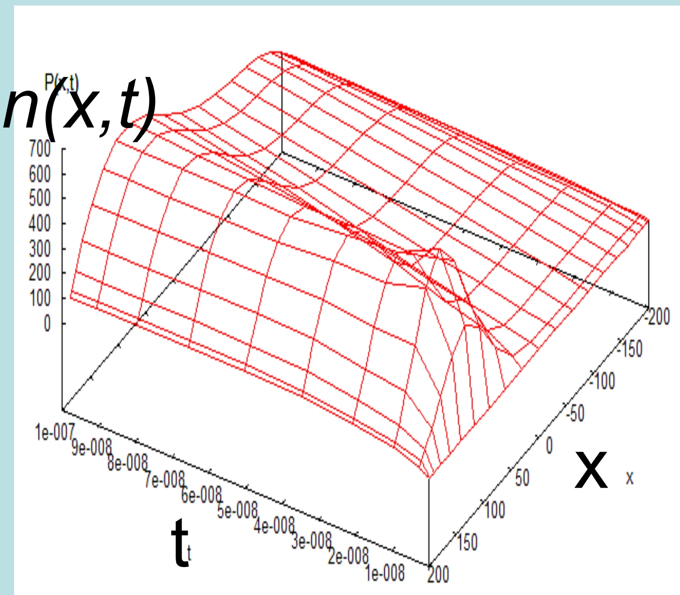
- We consider local injection of colder plasma, on top of a constant background loading (as before)

$$S(x,p,t) = S_B(x,p,t) + S_{0,I}\frac{1}{\sqrt{2\pi m_e k_B T}}\, e^{-\frac{p^2}{2m_e k_B T}}\, \frac{1}{\sqrt{2\pi}\sigma_I}\, e^{-\frac{(x-x_I)^2}{2\sigma_I^2}}\, (t_I - t)/t_I$$

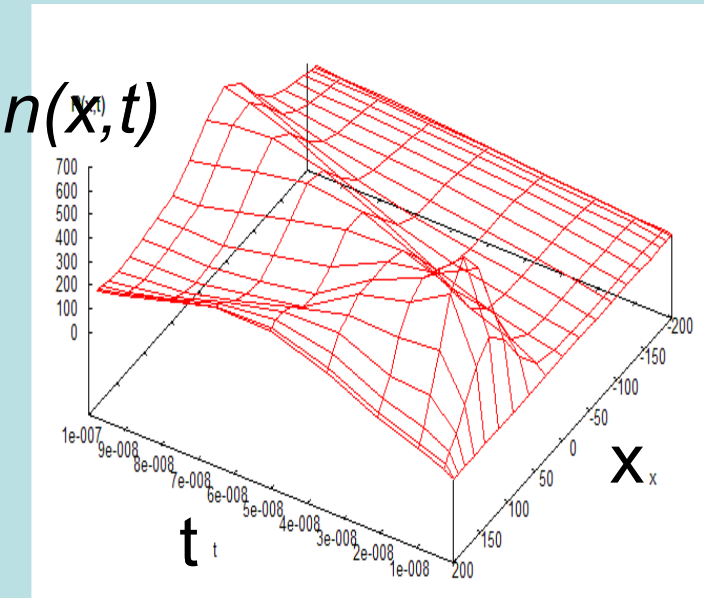
- The local source is Gaussian distributed in *p* (thermal), $T_I = T_B/10$ (lower temperature), and of Gaussian shape in the *x-direction* (localized injection),
  The injection lasts just for a short time at the beginning (for 1/20 of the total time shown).
  (T_I[keV] = 0.8 keV, $S_{0,I} = 10^{-1}$, $\sigma_I = 10^{-1}$, $x_I = ½\, r_2 = +100$ cm)



$q_x(\Delta x)$ Gaussian	$q_x(\Delta x) \sim \Delta x^{-1.2}$



Normal diffusion, just spreading of the density



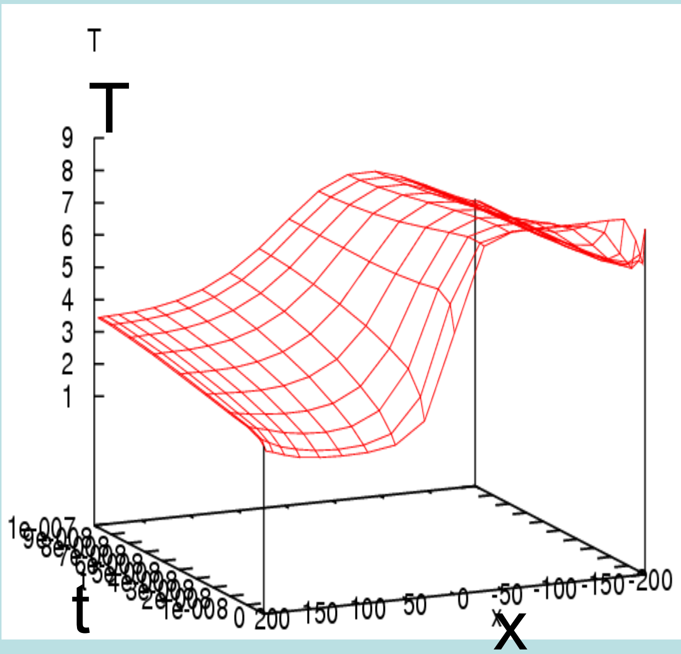
Super-diffusion, peak position $x_P \sim t \Rightarrow \langle x^2(t)\rangle \sim t^2$
$\Rightarrow$ fast inward transport

Temperature evolution

•The temperature distribution $T(x,t)$ can be determined from $P(x,p,t)$ by assuming a Gaussian shape in the p-direction (the Δp's are Gaussian distributed!), so that, for fixed x, t, one can determine the variance $\sigma_p^2(x,t)$ in the p-direction, and it holds that

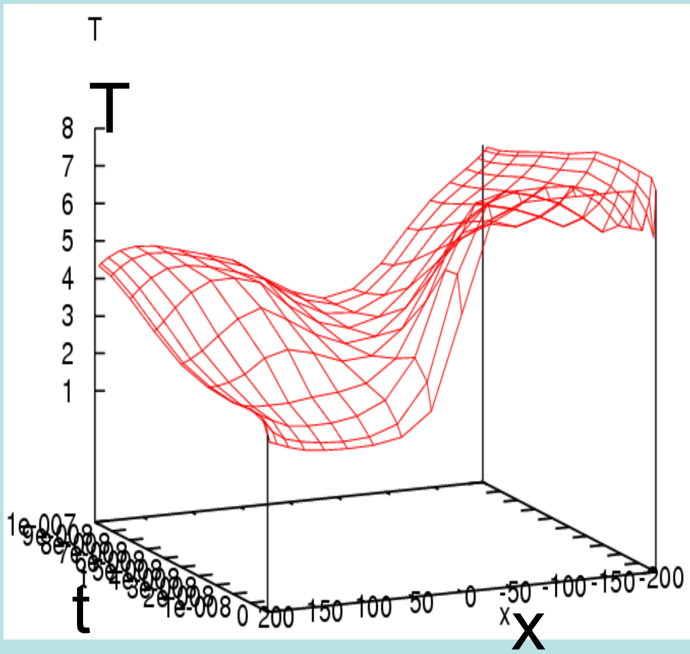
$$m_e\, k_B\, T(x,t) = \sigma_p^2(x,t).$$

$q_x(\Delta x)$ Gaussian



Slow heating of the injected plasmas, normal heat diffusion

$q_x(\Delta x) \sim \Delta x^{-1.2}$



Temperature drop moves inwards; fast inward propagation of a cold pulse, again $\langle x^2(t)\rangle \sim t^2$; faster heating of injection region than in normal diffusion

Conclusion

- The combined CTRW in position and momentum space was introduced, together with a method to solve it numerically.
- First results show that fast, anomalous transport phenomena arise naturally in the context of the model, e.g. fast inward propagation of a cold pulse. In the future, we will explore further phenomena, such as off axis heating.
- So-far, momentum space was treated classically, only classical diffusion was allowed (i.e. heating) in the *p*-direction – power-law distributed increments for the momentum will allow to study phenomena like particle acceleration.
- The connection of the used probabilities for the increments to experiments/observations or other turbulent transport models (e.g. fluid or MHD simulations, self-organized criticality (SOC) models, etc.) is to be worked out.