Magnetohydrodynamic Turbulence

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Turbulence

Turbulent flows: ensemble of random fluctuations without apparent structure/order

Systems appears to be 'smooth' (no specific feature/symmetry to cling to).

Under idealized conditions (statistical stationarity/homogeneity, no boundaries, no friction)

$\longrightarrow$(generalized) scale-covariance

Self-similar function $f(\ell) = A \cdot \ell^{\gamma} \longrightarrow f(\lambda\ell) \sim \lambda^{\gamma} f(\ell)$
Function $f(\ell)$ under magnifying glass ($\ell \to \lambda\ell$) looks identical (neglecting constant factor)

For simplificity: statistical isotropy, i.e. ensemble average $\langle \bullet \rangle$ independent of direction
implies stat. homogeneity (independence of position).

Turbulent fields exhibit statistical (self-)similarity !

Tackling the problem

Starting point for mostly phenomenological theories dealing with

- temporal/spectral evolution of low-order statistical moments, e.g. magnetic and kinetic energies, helicities, associated spectral fluxes
- spatially intermittent structure of turbulent fields

New development (emerging from turbulent passive-scalar transport):

- Lagrangian statistics and invariants

Applications:

- lifetime of/structure formation in interstellar molecular clouds (star-formation)
- transport/dispersion/acceleration of substances/particles (nuclear fusion/environmental sciences/cosmic rays)
- magnetic field amplification (turbulent dynamo)/formation of large-scale structures (meteorology)
- friction/mixing/flow control (engineering)

Ideal Invariants and Cascade directions

- total energy $E = \int_V dV (v^2 + b^2)$ — no dissipation
- cross helicity $H^{\mathrm{C}} = \int_V dV \mathbf{v} \cdot \mathbf{b}$ — frozen-in field lines
- magnetic helicity $H^{\mathrm{M}} = \int_V dV \mathbf{a} \cdot \mathbf{b}, \quad \mathbf{b} = \nabla \times \mathbf{a}$ — no reconnection

Ideal invariants satisfy detailed balance relations, e.g., triad interactions (quadratic nonlinearities)

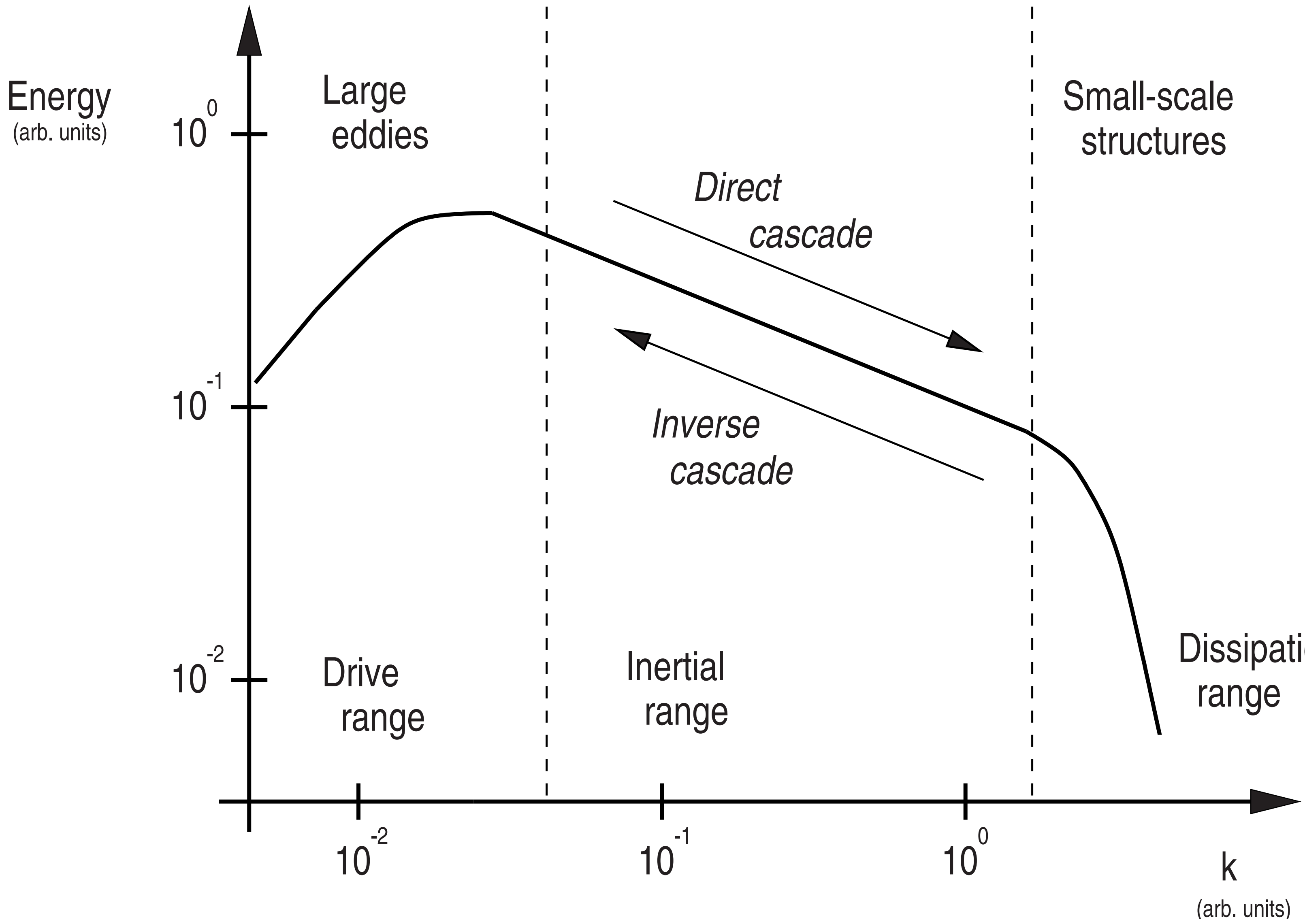
$$\dot{E}_{k_1} + \dot{E}_{k_2} + \dot{E}_{k_3} = 0, \quad \mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 = 0$$

$$\text{Inverse cascade} \quad \Longleftarrow_{\text{small } k} \qquad \Longrightarrow_{\text{large } k} \quad \text{direct cascade}$$

inverse cascade: formation of large-scale coherent structures.

Detailed balance prerequisite for cascade/power-law scaling.

Kolmogorov-Richardson Picture



Energy Cascade Phenomenology

- Kolmogorov (K41)

  Turbulent eddies break up in successively smaller structures

  Time-scale: $\tau_{\mathrm{NL}} \sim \ell / v_\ell, \quad \varepsilon \sim v_l^2 / \tau_{\mathrm{NL}}, \quad v_\ell^2 \sim k E_k$

  $\longrightarrow$ Energy spectrum $E(k) \sim k^{-5/3}$

- Iroshnikov-Kraichnan (IK)

  Alfvén waves interact nonlinearly along magnetic field

  Time-scale: $\tau_{\mathrm{A}} \sim \ell / B_0, \quad \varepsilon \sim v_l^2 / \tau_*, \quad \tau_* \sim \frac{\tau_{\mathrm{NL}}}{\tau_{\mathrm{A}}} \tau_{\mathrm{NL}}$

  $\longrightarrow$ Energy spectrum $E(k) \sim k^{-3/2}$

- Goldreich-Sridhar

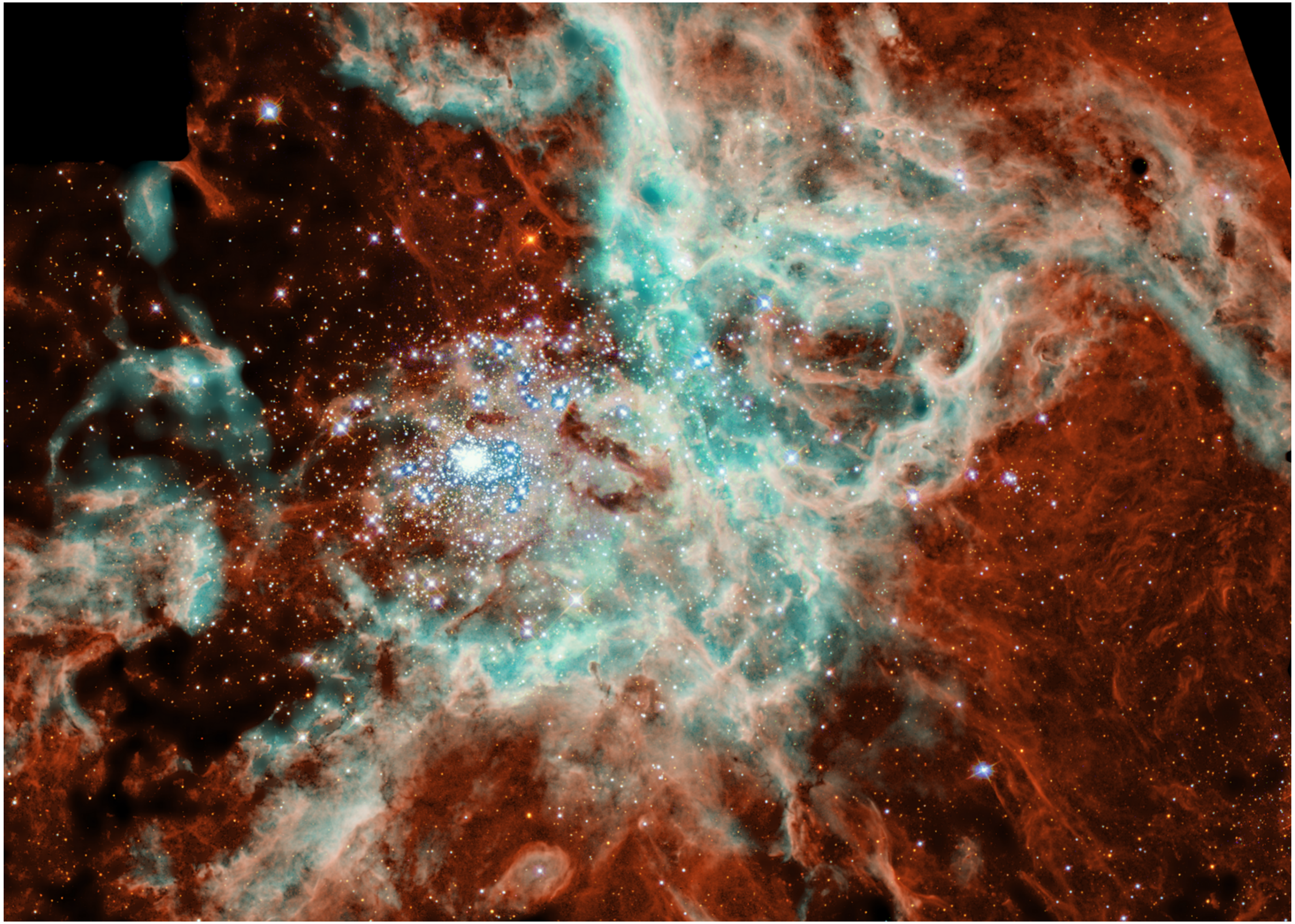
  Magnetic field causes local anisotropy

  $\longrightarrow$ Field-parallel: transfer negligible

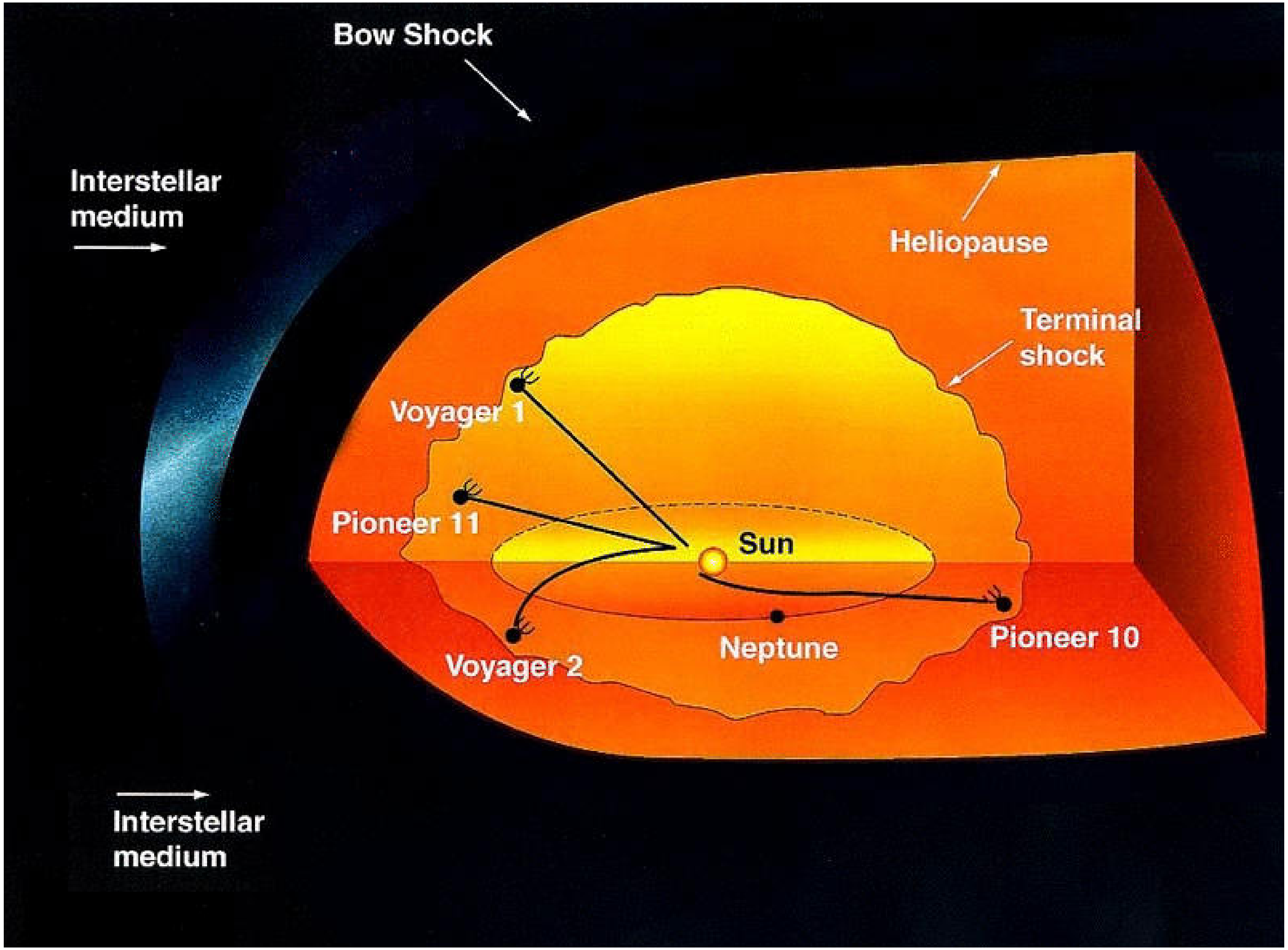
  $\longrightarrow$ Field-perpendicular: Kolmogorov cascade

  $\longrightarrow$ Perpendicular energy spectrum $E(k_\perp) \sim k_\perp^{-5/3}$

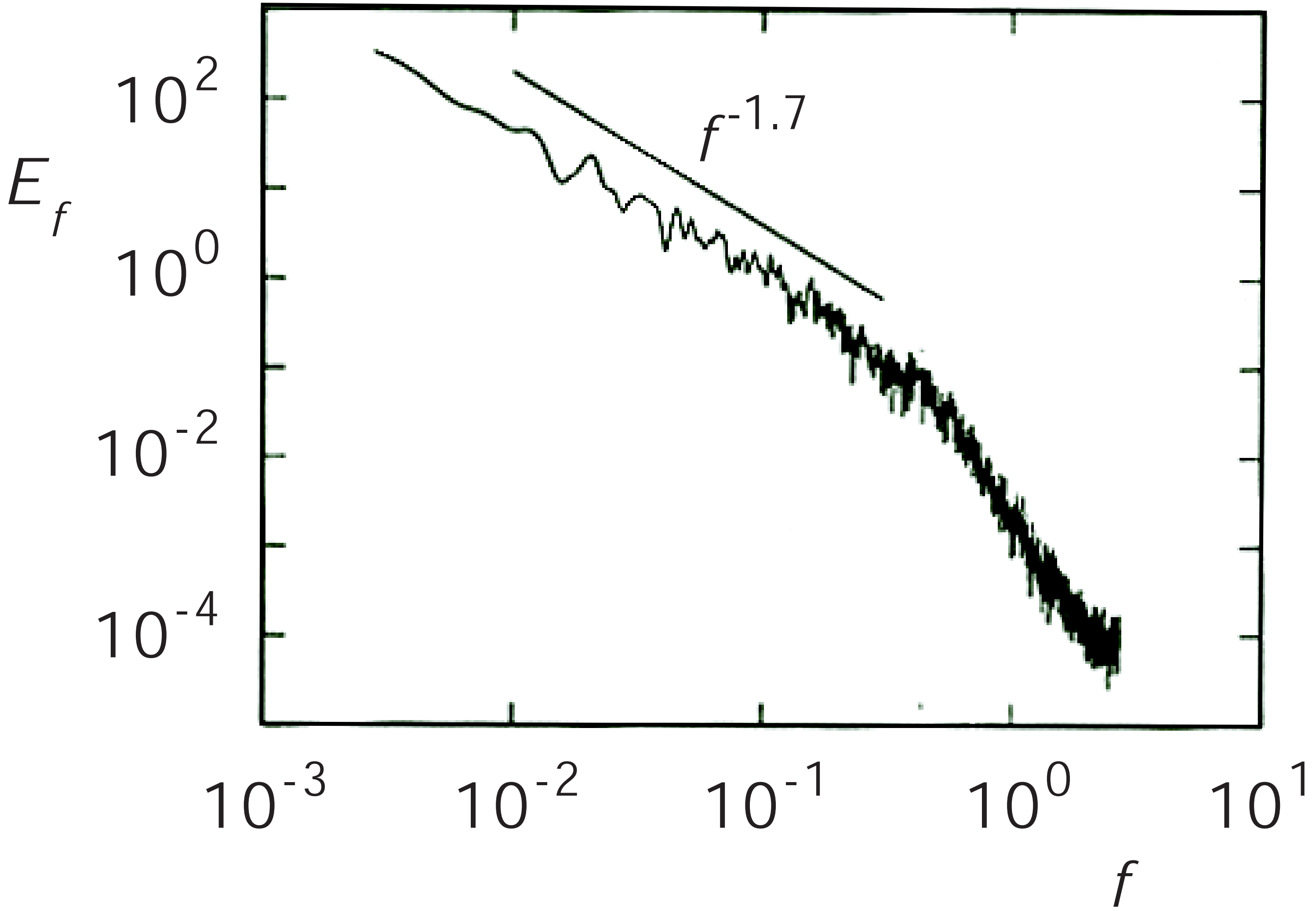
Doradus 30



Probing the Solar Wind



Experimental Observation



Leamon et al. JGR ’98

Solar wind fluctuations measured by WIND probe at $\simeq 1 A.U. \Rightarrow$ K41 scaling $\sim k^{-5/3}$

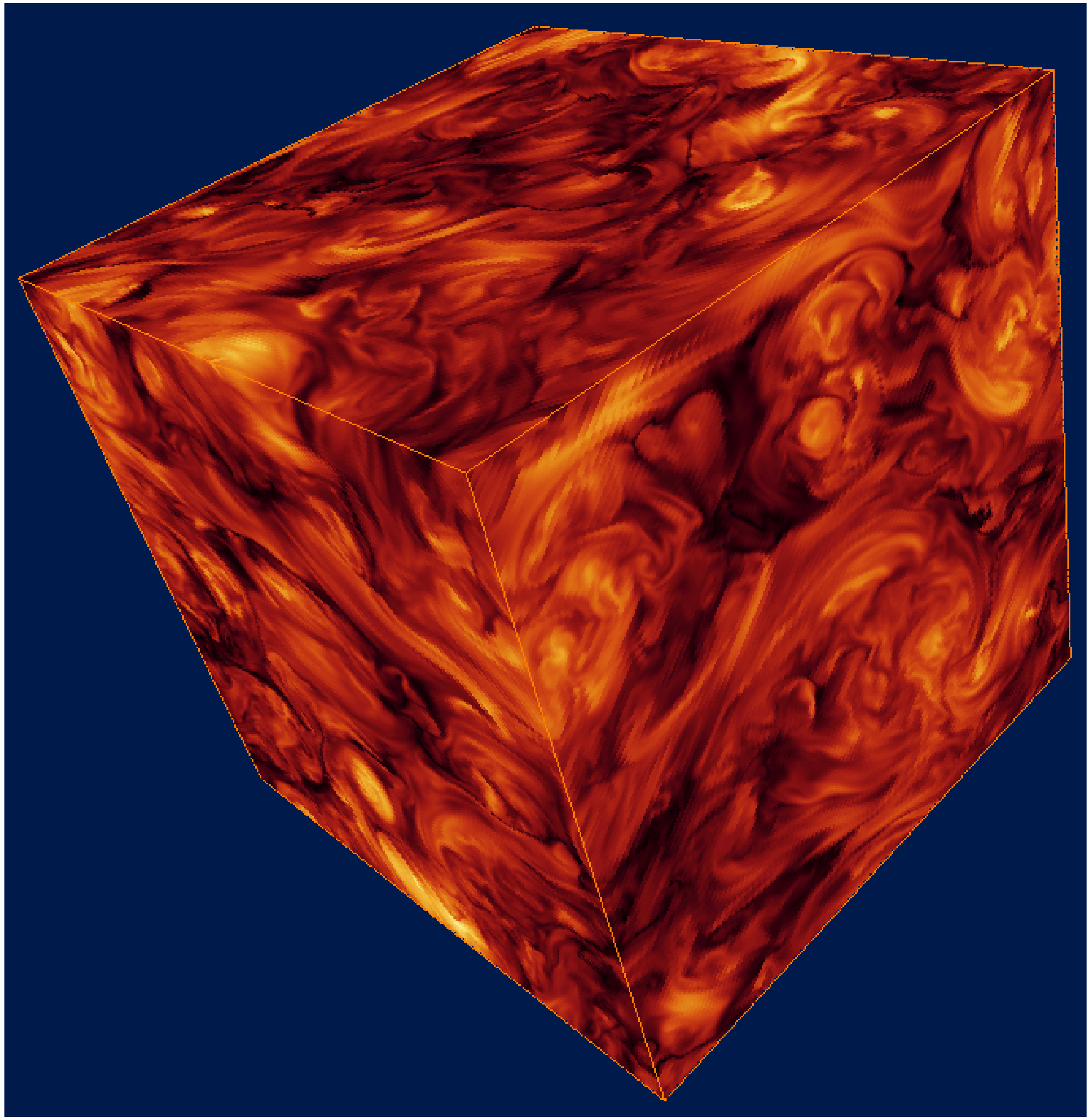
Incompressible Magnetohydrodynamics (MHD)

Simplified incompressible fluid model:

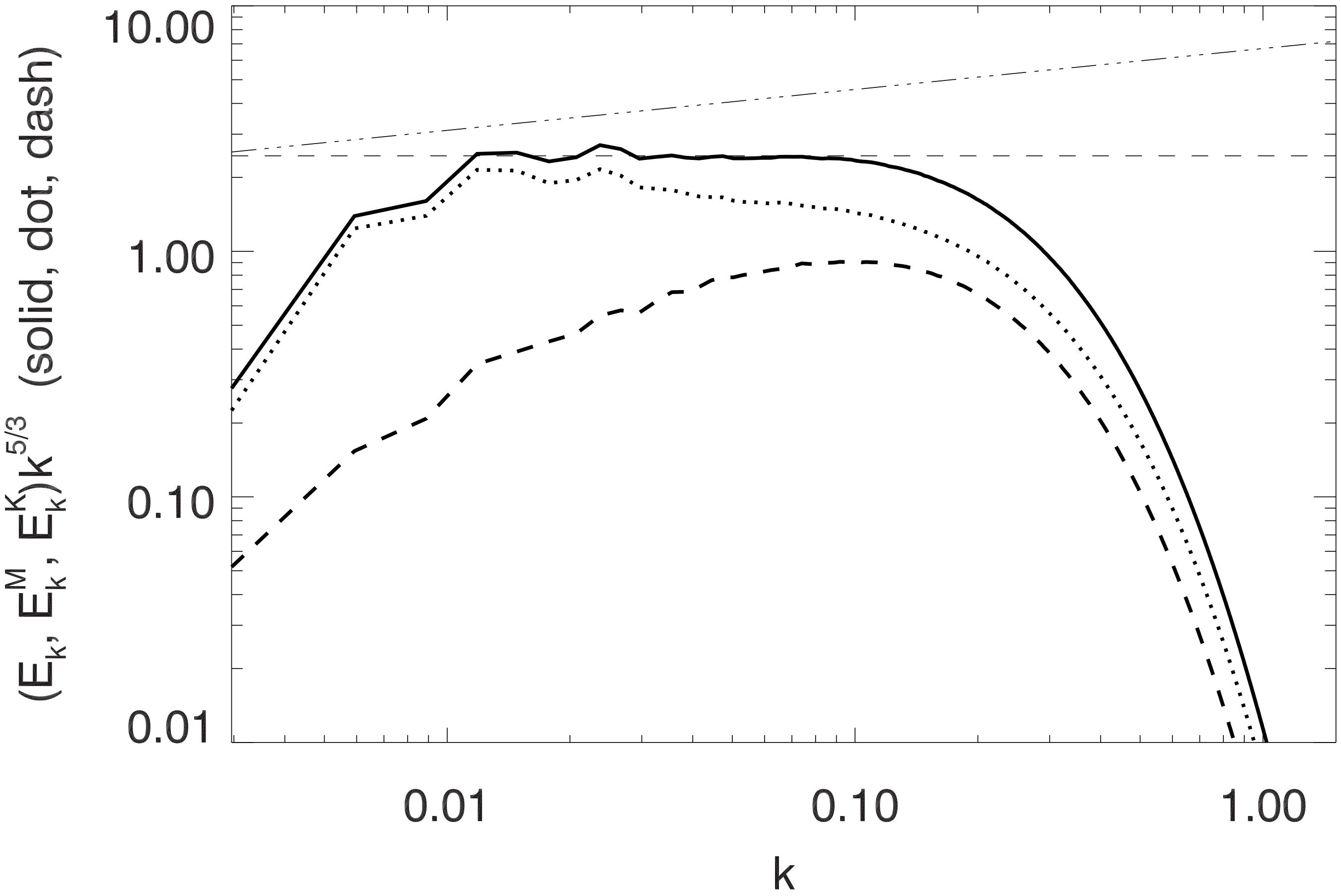
$$\begin{aligned}\partial_t \mathbf{v} =& -(\mathbf{v}\cdot\nabla)\mathbf{v} - \nabla p - \mathbf{b}\times(\nabla\times\mathbf{b}) + \mathrm{Re}^{-1}\Delta\mathbf{v}\,,\\ \partial_t \mathbf{b} =& \nabla\times(\mathbf{v}\times\mathbf{b}) + \mathrm{Rm}^{-1}\Delta\mathbf{b}\,,\\ \nabla\cdot\mathbf{v} =& \nabla\cdot\mathbf{b} = 0\,.\end{aligned}$$

- Kinetic and magnetic Reynolds number: $\mathrm{Re} := \frac{\ell_0 v_0}{\mu}$ $\quad \mathrm{Rm} := \frac{\ell_0 v_0}{\eta}$
- Kinematic viscosity μ, magnetic diffusivity η
- Turbulence, if Re,Rm$\gg 1$
  - Solar convection zone (Re$\sim 10^{15}$, Rm$\sim 10^{8}$)
  - Black hole accretion disk (Re$\sim 10^{11}$, Rm$\sim 10^{10}$)
  - Earth's liquid core (Re$\sim 10^{9}$, Rm$\sim 10^{2}$)

Turbulent Magnetic Field (Isotropic)



Numerical Simulation (Isotropic)

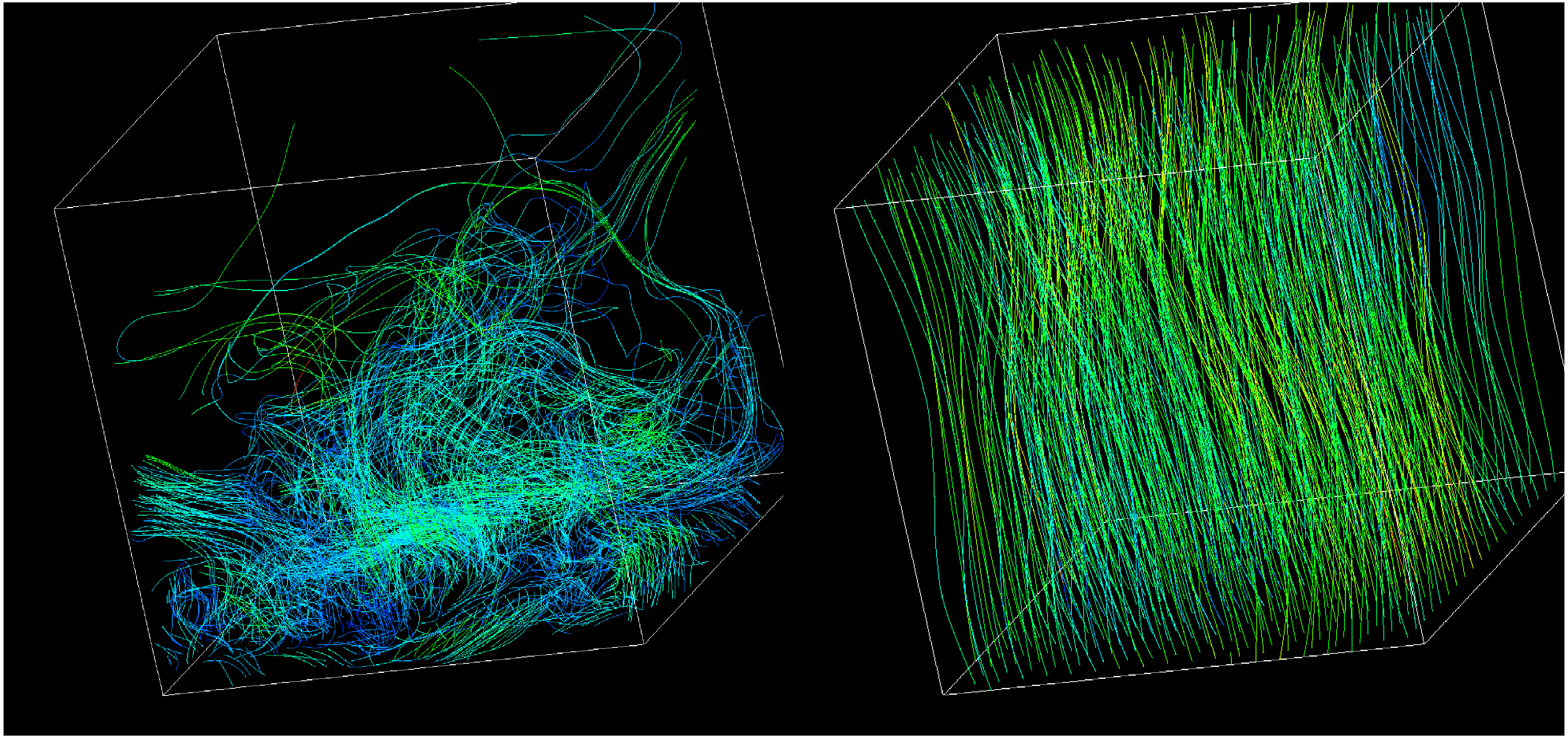


Pseudospectral direct numerical simulation (1024^3 collocation points)

Three-dimensional periodic cube

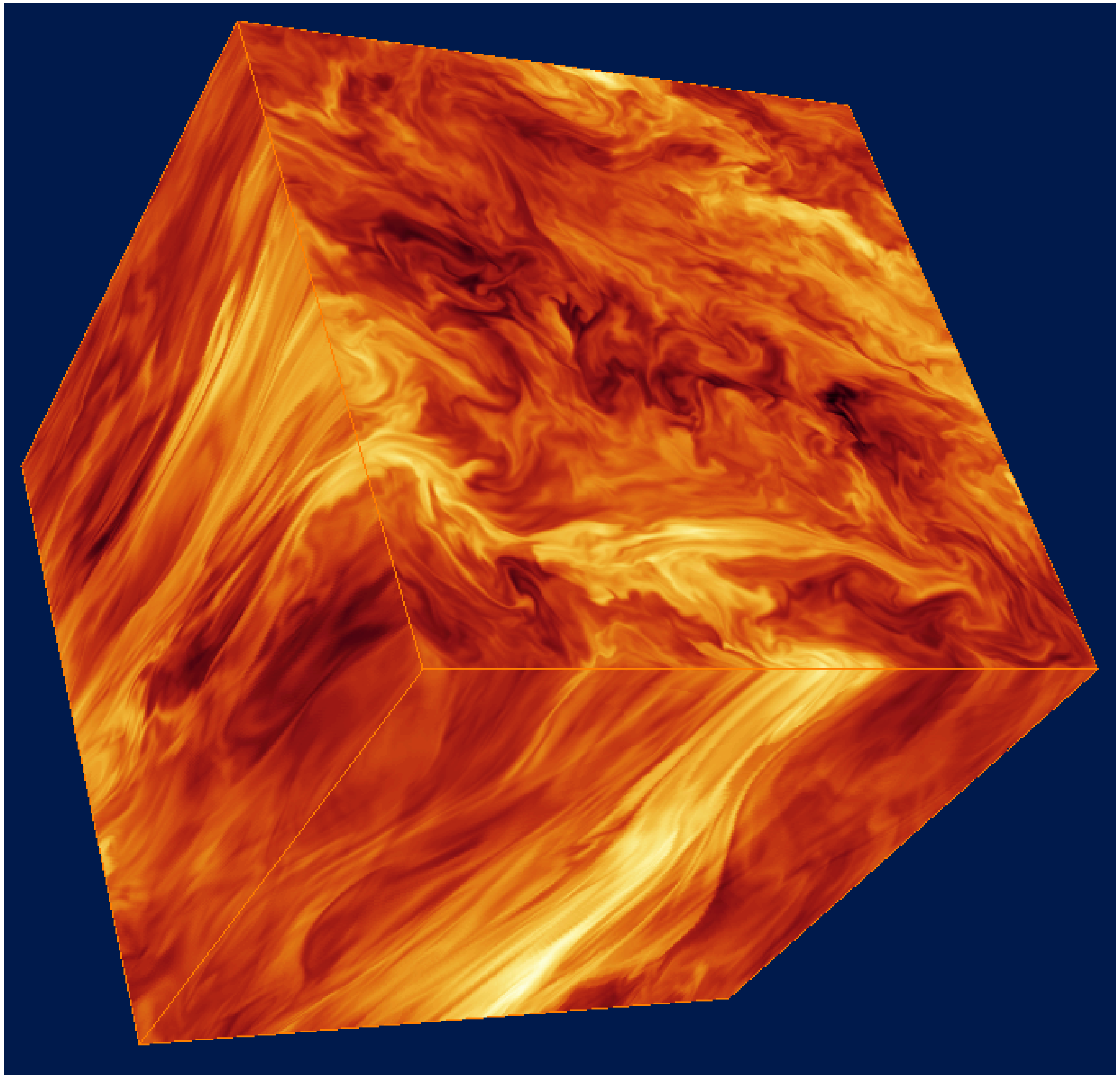
Initially: nonhelical isotropic random fields with amplitudes $\sim \exp[-k^2/(2k_0^2)]$, $k_0 = 4$

Introducing Anisotropy

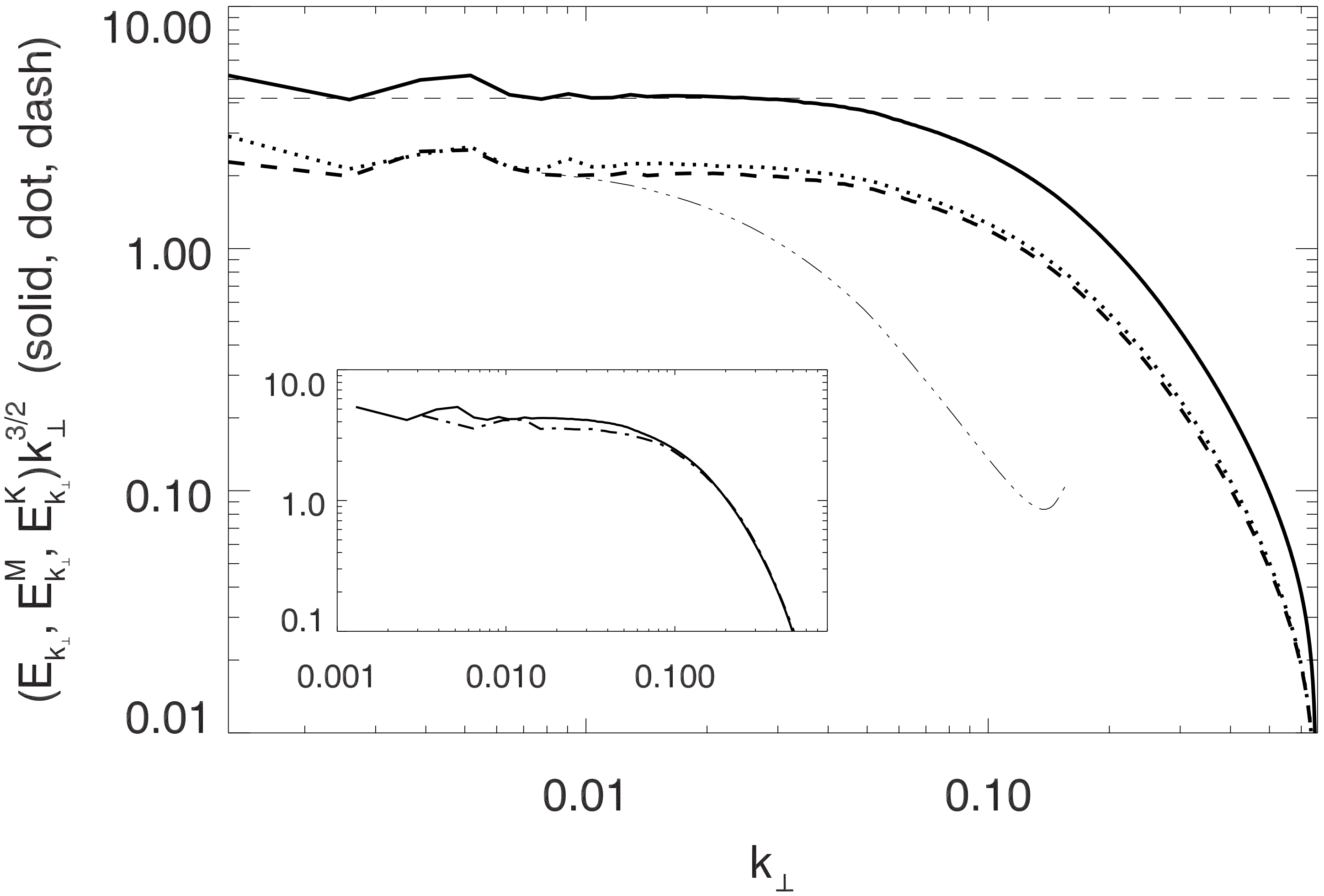


Switching from isotropic K41 to anisotropic Goldreich-Sridhar configuration
by imposed mean magnetic field $\mathbf{B}_0 = B_0\mathbf{e}_z$ ($B_0 \simeq 5|\mathbf{b}|_{\mathrm{rms}}$)

Turbulent Magnetic Field (Anisotropic)



Numerical Simulation (Anisotropic)



Three-dimensional forced anisotropic turbulence ($1024^2 \times 256$ collocation points) displays IK-scaling $\sim k^{-3/2}$

Closure Theory

Regarding statistical moments of fluid equations schematically:

$$\partial_t \langle u \rangle = \langle uu \rangle$$
$$\partial_t \langle uu \rangle = \langle uuu \rangle$$
$$\partial_t \langle uuu \rangle = \langle uuuu \rangle$$
$$\vdots$$

Closure (Quasi-normal approximation):
4th and higher order moments $\longrightarrow$ Expressed via second-order moments

Problem: Unphysical, negative energy spectra possible
Solution: Introduction of damping term on 3rd order level
(Eddy-damped-quasi-normal-Markovian (EDQNM) approximation)

Spectral EDQNM Equations

Equation for energy spectrum E_k:

$$(\partial_t + 2\mathrm{Re}^{-1}k^2)E_k = \int\!\!\int_{\triangle} \mathrm{d}p\,\mathrm{d}q\,\Theta_{kpq}\,T_{kpq}$$

- '$\triangle$': Integration over modes with $\mathbf{k}+\mathbf{p}+\mathbf{q}=0$
- $T_{kpq} = T_{kpq}(E_p, E_q, \ldots)$ complicated energy transfer function
- Θ_{kpq} phenomenological relaxation time of triad interactions
  (remains of Green's function after Markovianization)

Inertial range: Constant spectral energy flow ε towards small-scales (direct cascade)

$$\partial_t E = \varepsilon = \int\int\int \mathrm{d}k\,\mathrm{d}p\,\mathrm{d}q\,\Theta_{kpq}T_{kpq} \sim \Theta_k k^4 E_k^2$$

With $\Theta_k = \left(\tau_{\mathrm{NL}}^{-1} + \tau_{\mathrm{A}}^{-1}\right)^{-1} \Rightarrow$ Quartic equation in E_k

$$\left.\begin{array}{ll}\tau_{\mathrm{NL}} \ll \tau_{\mathrm{A}} \Rightarrow E_k \sim k^{-5/3} & \mathrm{K41} \\ \tau_{\mathrm{A}} \ll \tau_{\mathrm{NL}} \Rightarrow E_k \sim k^{-3/2} & \mathrm{IK}\end{array}\right\} \text{Phenomenological dead-end}$$

Matthaeus & Zhou, Phys.Fluids B, '89

Inertial-Range Energetics

EDQNM equation for residual energy spectrum, $E_k^{\mathrm{R}} = E_k^{\mathrm{M}} - E_k^{\mathrm{K}}$:

$$(\partial_t + 2\mathrm{Re}^{-1}k^2)E_k^{\mathrm{R}} = \int\int_\triangle \mathrm{d}p\,\mathrm{d}q\,\Theta_{kpq}R_{kpq}$$

Right-hand side complicated function with two types of contributions:

- Spectrally local interactions ($k \sim p \sim q$):
  - fluid scrambling on time scale $\tau_{\mathrm{NL}} \sim \frac{\ell}{\sqrt{v_\ell^2 + b_\ell^2}} \sim (k^3 E_k)^{-1/2}$ (Dynamo effect)
  - $R^{\mathrm{Dyn}} \sim \Theta_k k^3 E_k^2$
- Spectrally non-local interactions (e.g. $k \ll p \sim q$):
  - Alfvén-wave scattering on time scale $\tau_{\mathrm{A}} \sim (kB_0)^{-1} \simeq (k^2 E^{\mathrm{M}})^{-1/2}$ (Alfvén effect)
  - $R^{\mathrm{Alf}} \sim \Theta_k k^2 E^{\mathrm{M}} E_k^{\mathrm{R}}$

Residual Energy

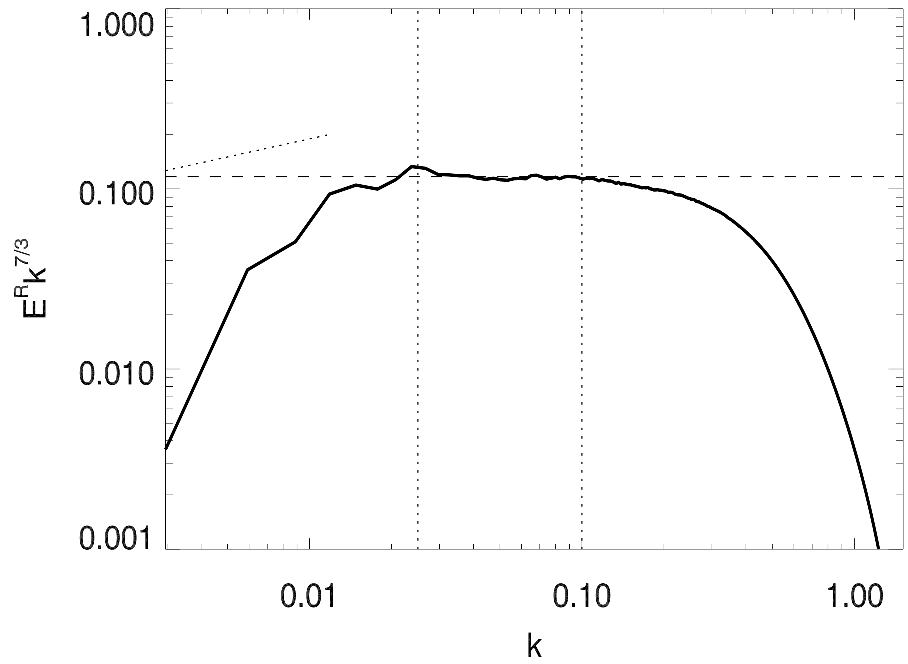
Assuming equilibrium between

— magnetic field amplification by field line streching (small-scale dynamo)

— energy equipartition by Alfvén wave effect

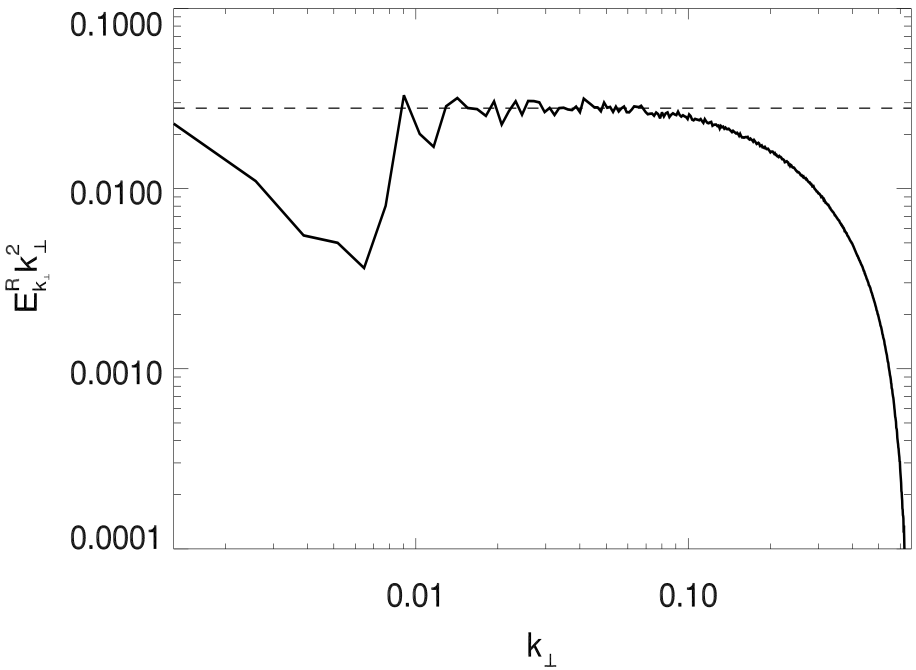
$$\Rightarrow E_k^{\mathrm{R}} \sim \left(\frac{\tau_{\mathrm{A}}}{\tau_{\mathrm{NL}}}\right)^2 E_k \sim k E_k^2$$

Isotropic 1024^3 simulation, $B_0 = 0$



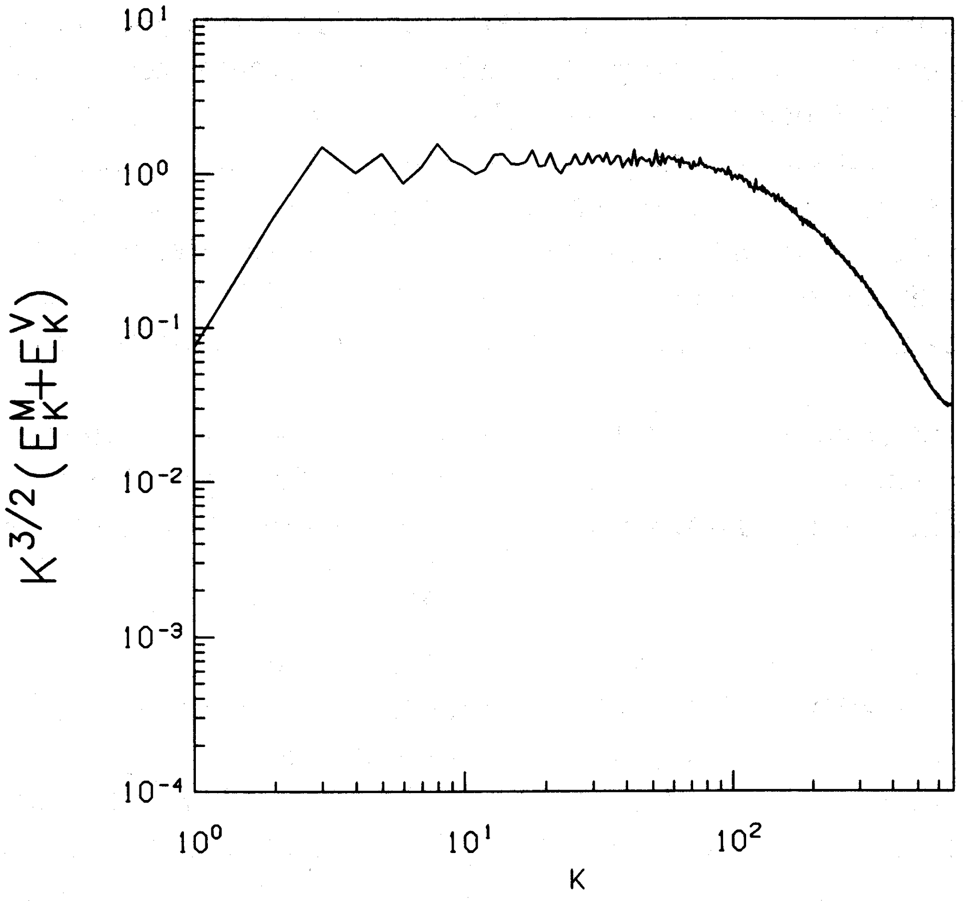
K41: $E_k \sim k^{-5/3} \Rightarrow E_k^{\mathrm{R}} \sim k^{-7/3}$

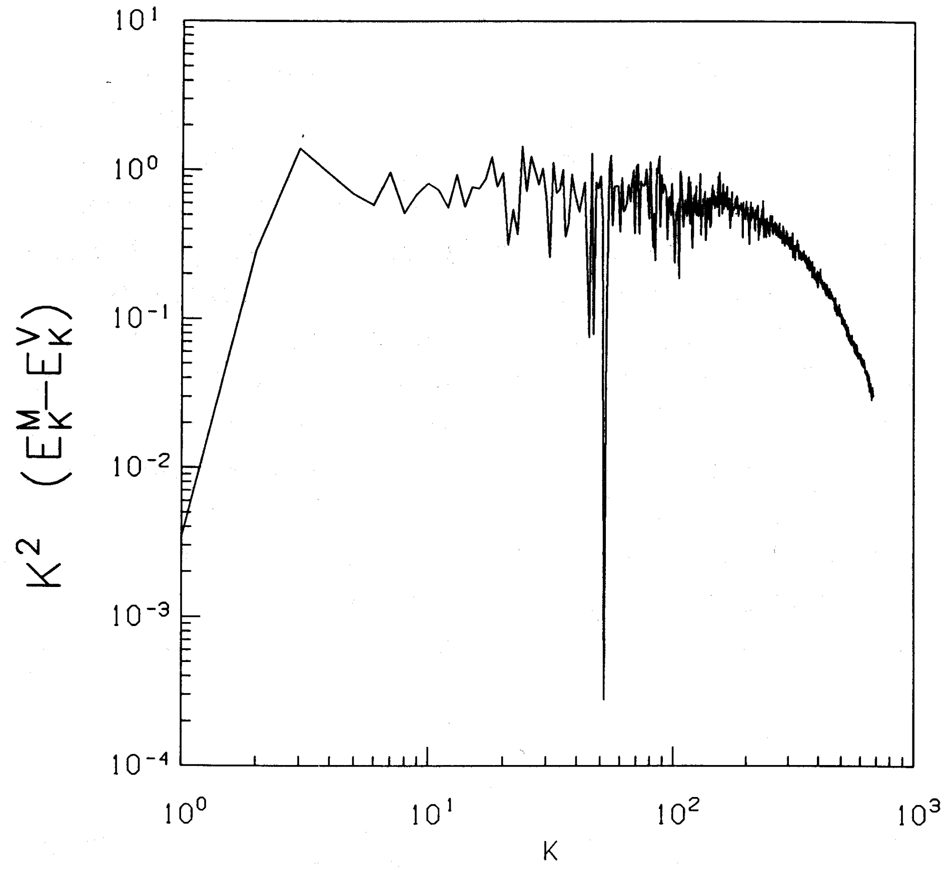
Anisotropic $1024^2 \times 256$ simulation, $B_0 = 5$



IK: $E_k \sim k^{-3/2} \Rightarrow E_k^{\mathrm{R}} \sim k^{-2}$

Two-Dimensional Simulations (MHD)





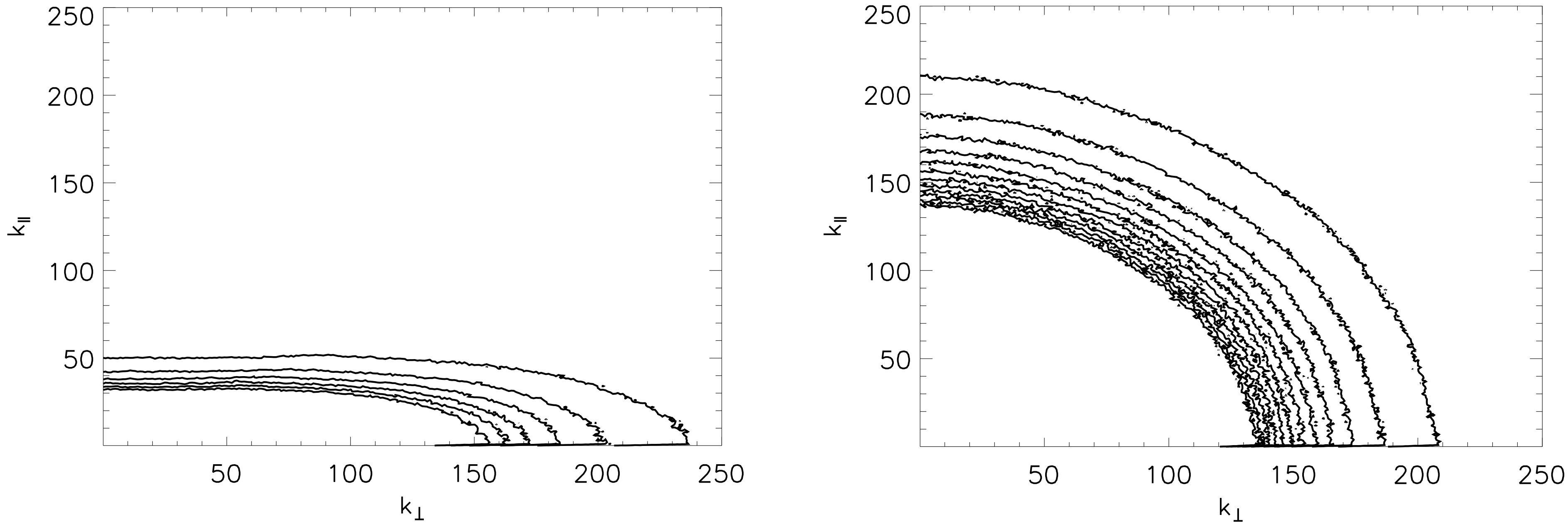
Left: Total energy spectrum $\times k^{3/2}$

Right: Residual energy spectrum $\times k^2$

2048^2 spectral MHD turbulence simulations

Biskamp & Schwartz Chaos, Solitons & Fractals ’91

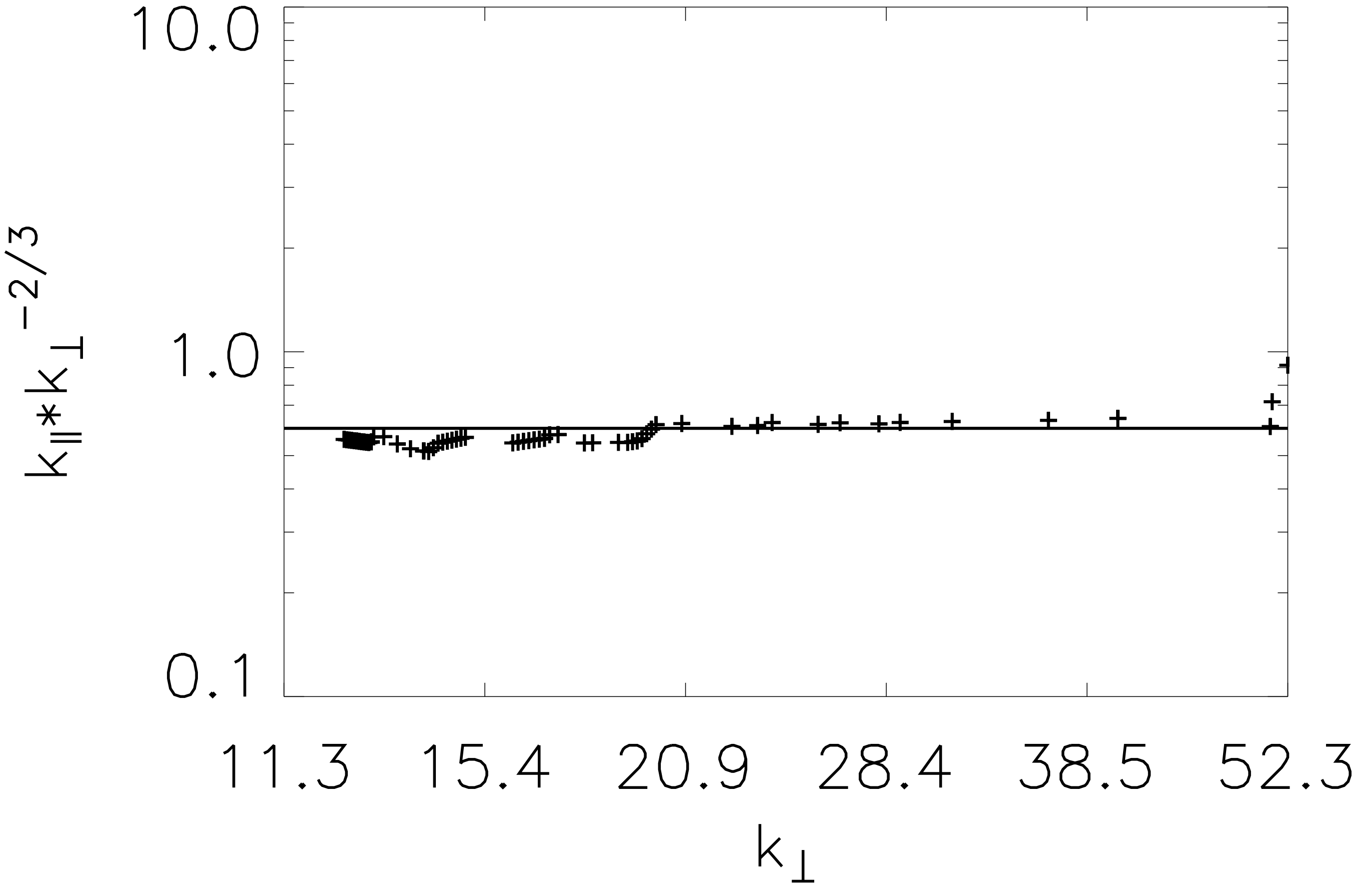
Energy Contours in Plane along $\mathbf{B}_0$



Strong anisotropy visible. As opposed to isotropic simulation (nearly perfect circles).

Cho & Vishniac ApJ, ’00

$k_\perp$-$k_\parallel$ Scaling



Consequence of $\tau_{NL} \sim \tau_A$ ('critical balance')

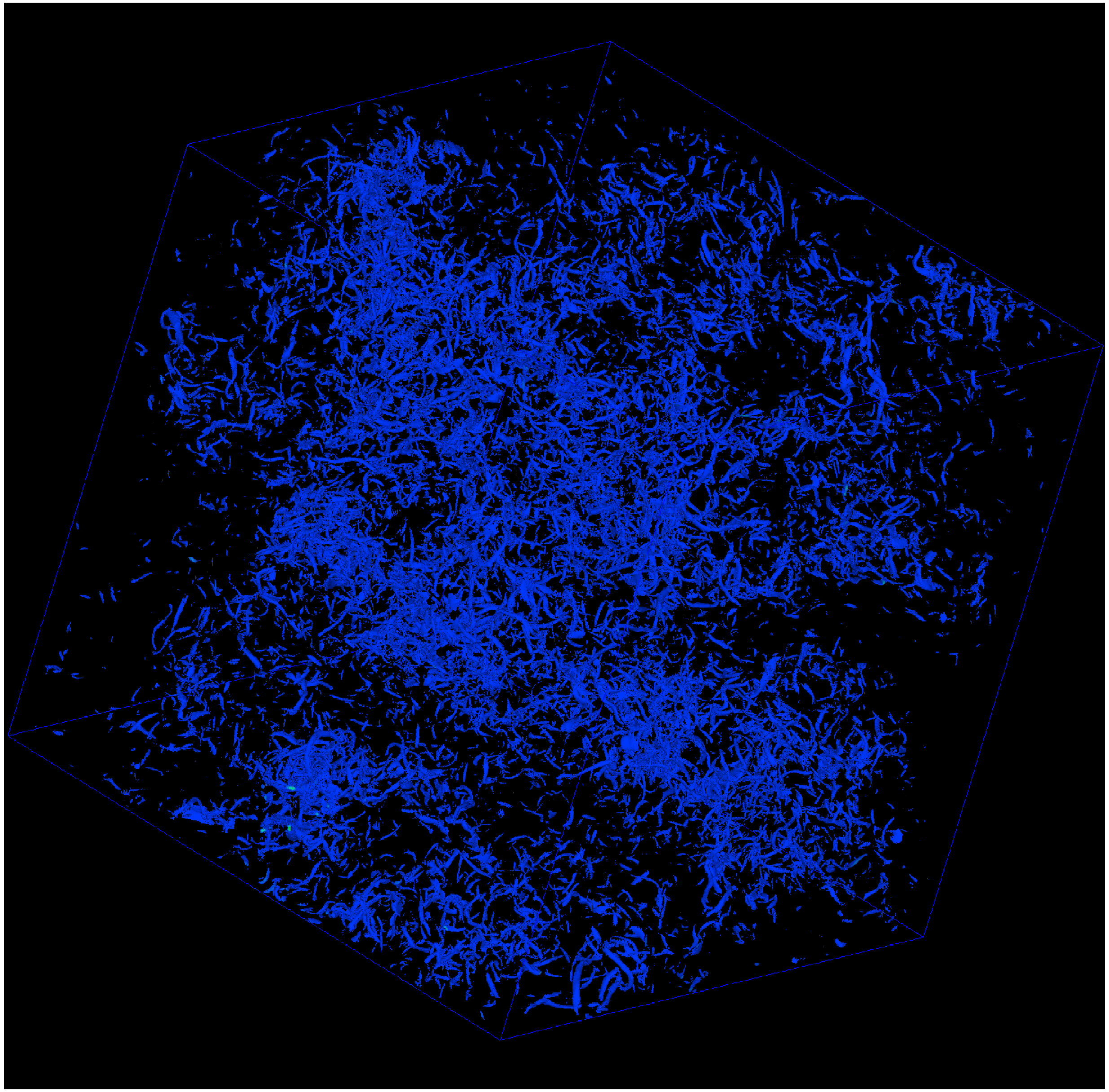
Distortion of field line by eddy of size ℓ on time-scale τ_{NL}

triggers Alfvén wave of length $\lambda \sim b_0 \tau_A$

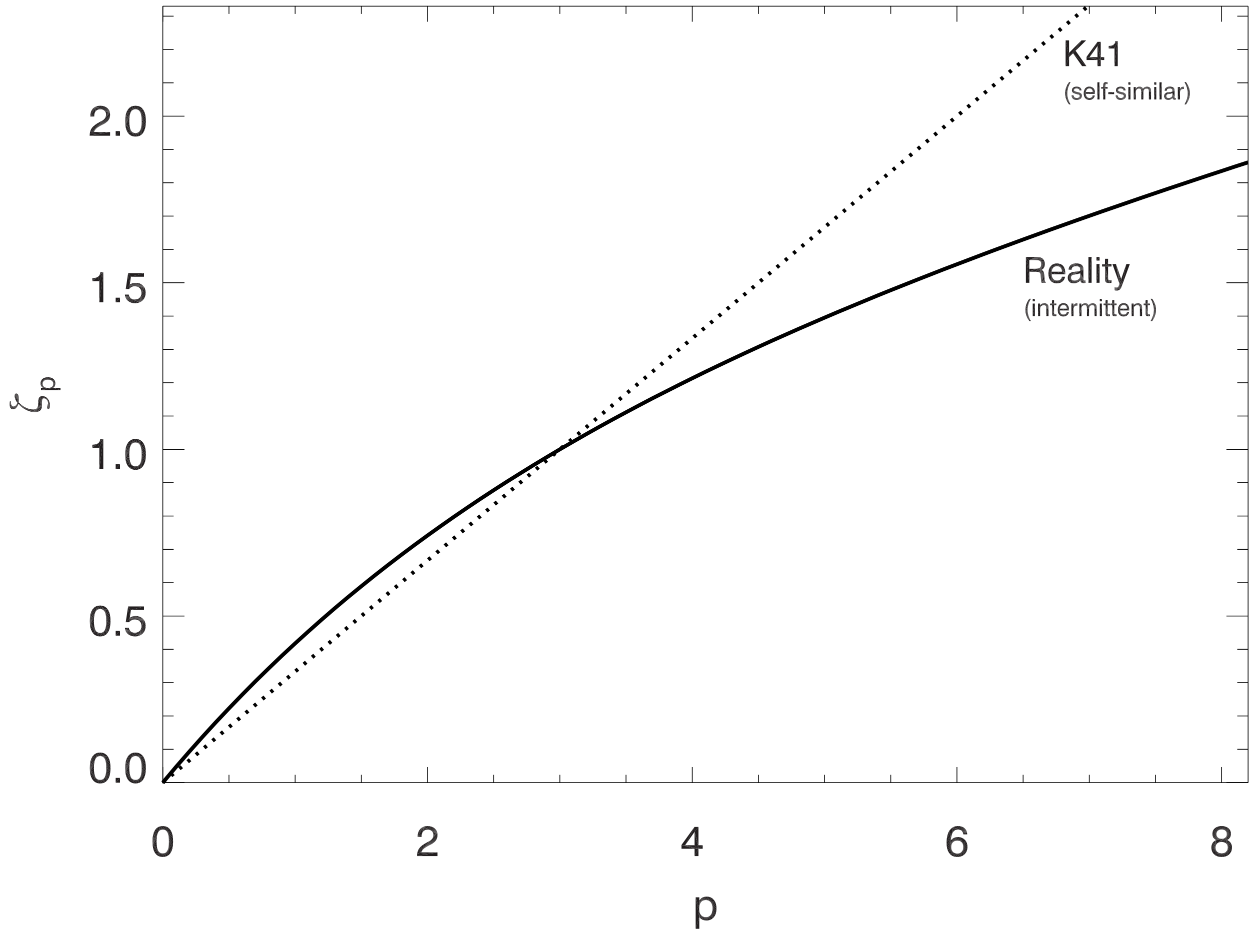
$$\Rightarrow k_\parallel \sim k_\perp^{2/3}$$

Goldreich & Sridhar ApJ '94, Galtier et al. '05

Spatial Structure of Dissipation (Hydrodynamics)

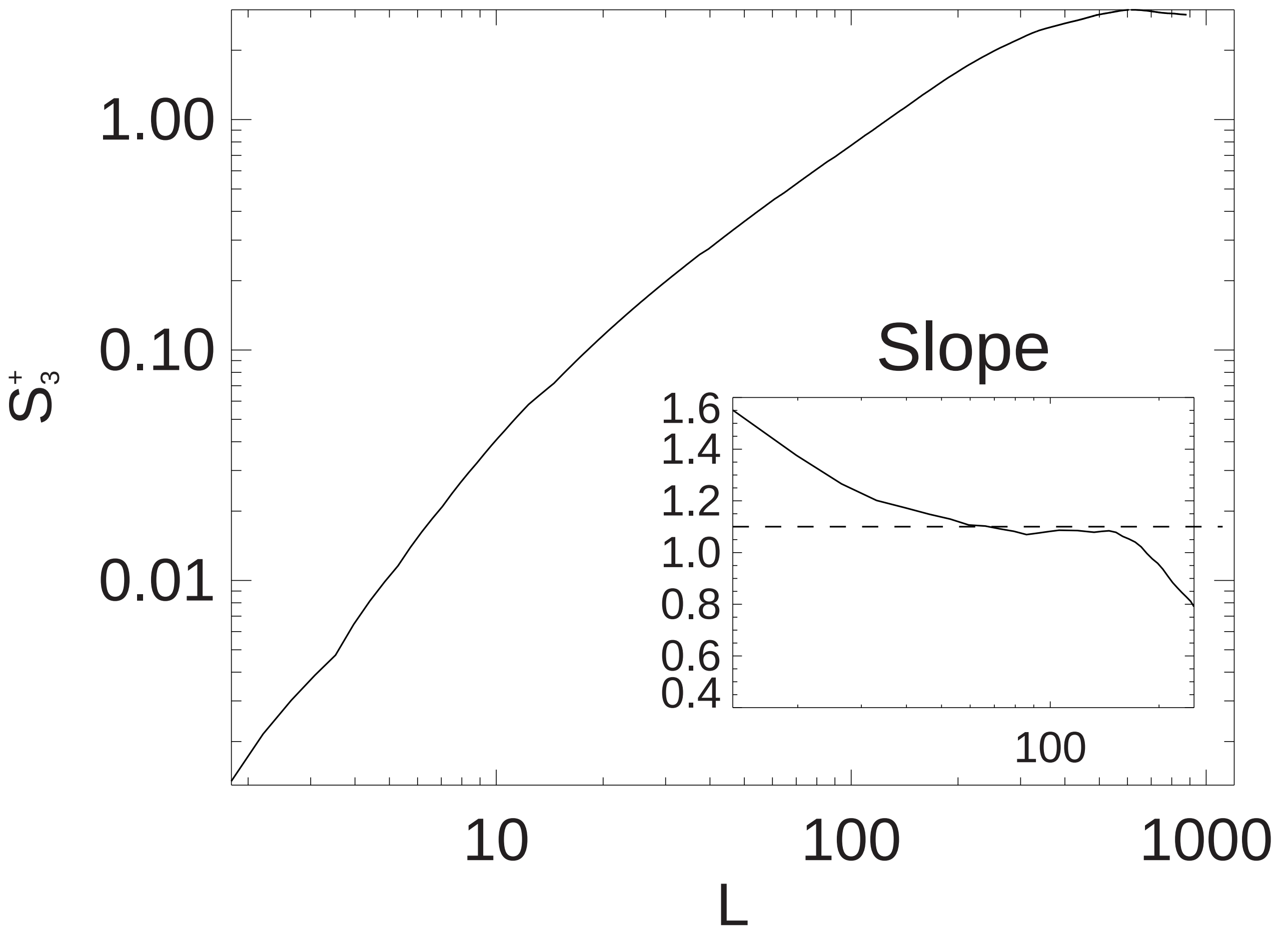


Measuring Structure



- Regard turbulent field difference over distance ℓ, $\delta v_\ell = [\mathbf{v}(\mathbf{x}) - \mathbf{v}(\mathbf{x} + \ell)] \cdot \hat{\ell}$
- Statistical moments $\langle \delta v_\ell^p \rangle \sim \ell^{\zeta_p}$ display power-law scaling
- Change of scaling exponents ζ_p indicates deviation from self-similarity

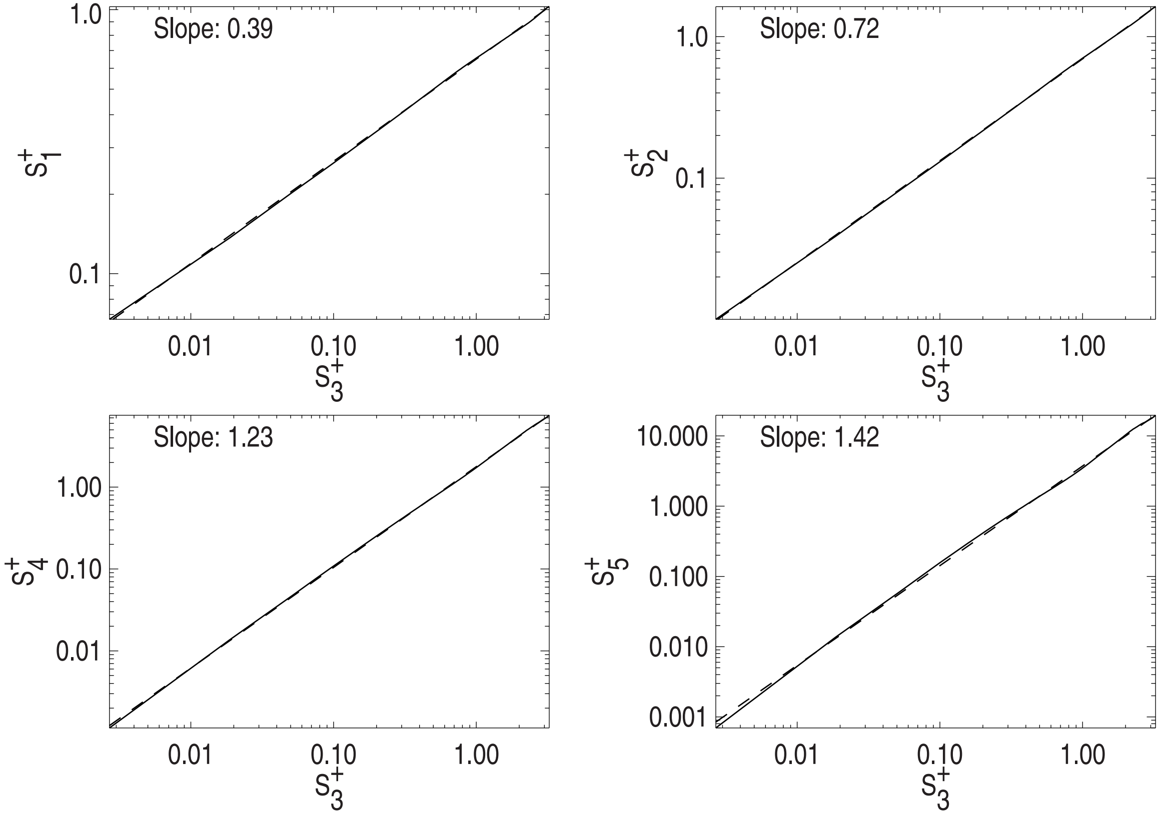
Third-Order Structure Function



Hydrodynamics: $S^3 = \frac{4}{5}\varepsilon\ell$ — Kolmogorov, '41

MHD: $\sum_{i=1}^{3}\langle\delta z_\ell^{\mp}(\delta_i z_\ell^{\pm})^2\rangle = -\frac{4}{3}\varepsilon^{\pm}\ell$ — Politano & Pouquet PRE & GRL '98

Extended Self-Similarity (ESS)

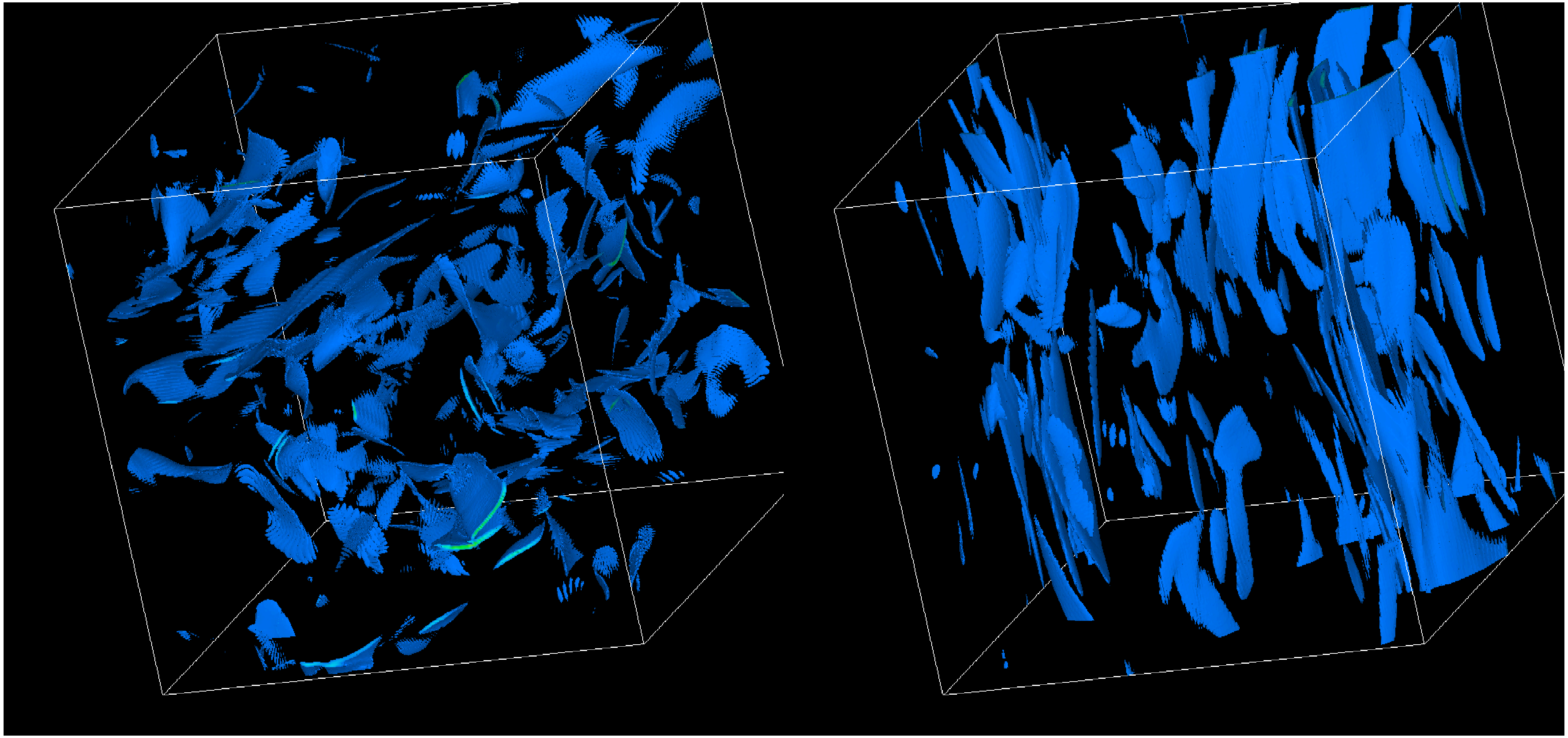


Observe extended scaling-range by plotting structure functions, $S_q \sim \ell^{\zeta_q}$, against reference structure function, $S_{q_0} \sim \ell^{\zeta_{q_0}}$:

$$\Rightarrow S_q(S_{q_0}) \sim \ell^{\zeta_q \zeta_{q_0}} \sim \ell^{\xi_q} \qquad \Rightarrow \zeta_q = \xi_q / \zeta_{q_0}$$

Benzi et al. PRE '93

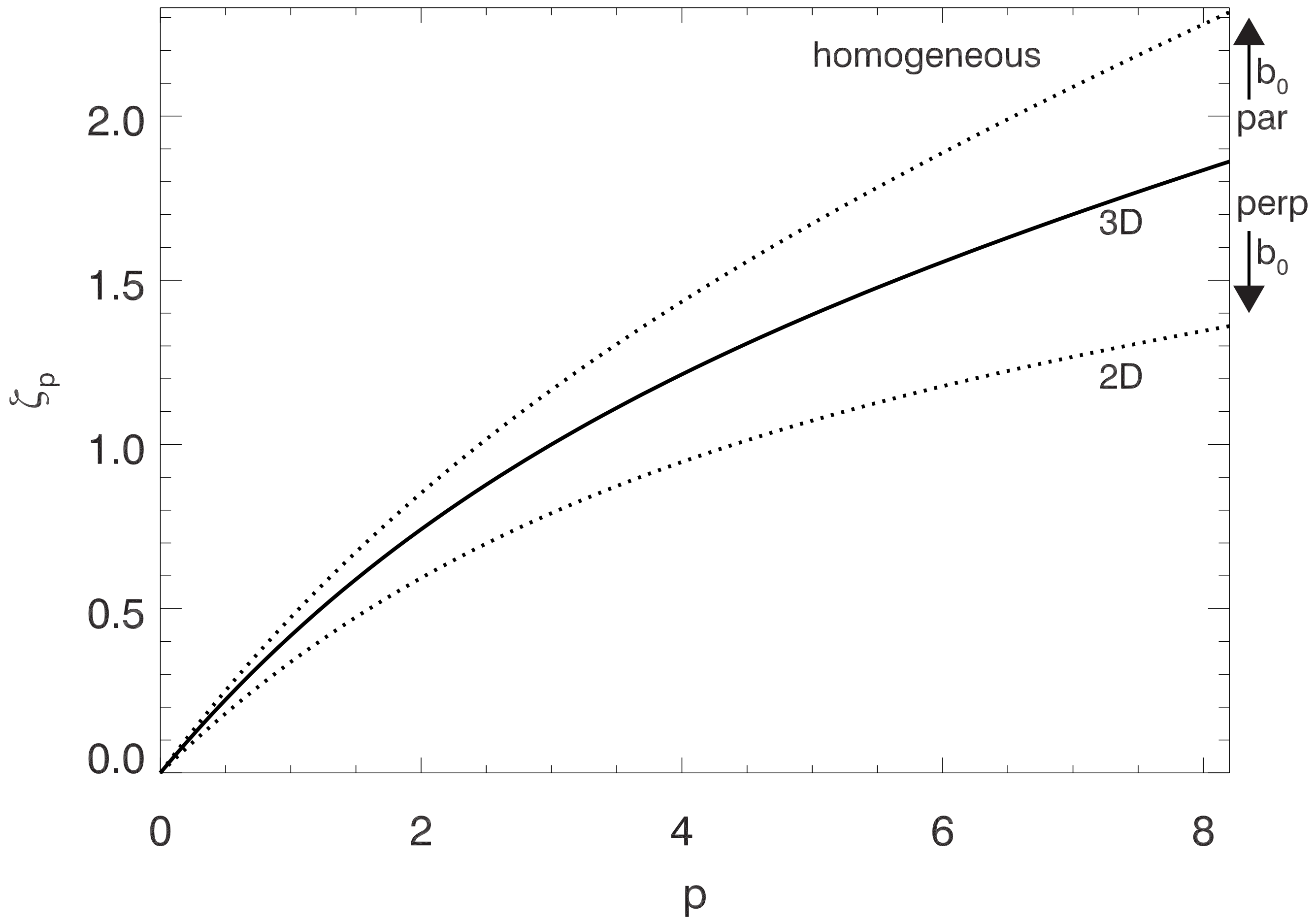
Spatial Structure of Dissipation (MHD)



Left: Dissipative current sheets in isotropic MHD turbulence

Right: Same picture with strong mean magnetic field pointing upwards

Intermittency Manipulation



- Taking differences parallel/perpendicular to $\mathbf{B}_0$ and varying field strength
- Parallel structure functions indicate asymptotically homogeneous fields
- Perpendicular structure functions show transition towards two-dimensionality

Log-Poisson Model

Regarding dissipative energy flux at scale ℓ, ε_ℓ
under refined similarity hypothesis $v_\ell \sim \ell^{\zeta_p}, \quad \langle \varepsilon_\ell^p \rangle \sim \ell^{\tau_p}$.
Assuming hierarchy

$$\varepsilon_\ell^{(p+1)}/\varepsilon_\ell^{(\infty)} \sim \left[\varepsilon_\ell^{(p)}/\varepsilon_\ell^{(\infty)}\right]^\beta, \qquad \varepsilon_\ell^{(p)} = \langle \varepsilon_\ell^{p+1} \rangle / \langle \varepsilon_\ell^p \rangle,\ \beta \in [0,1]$$

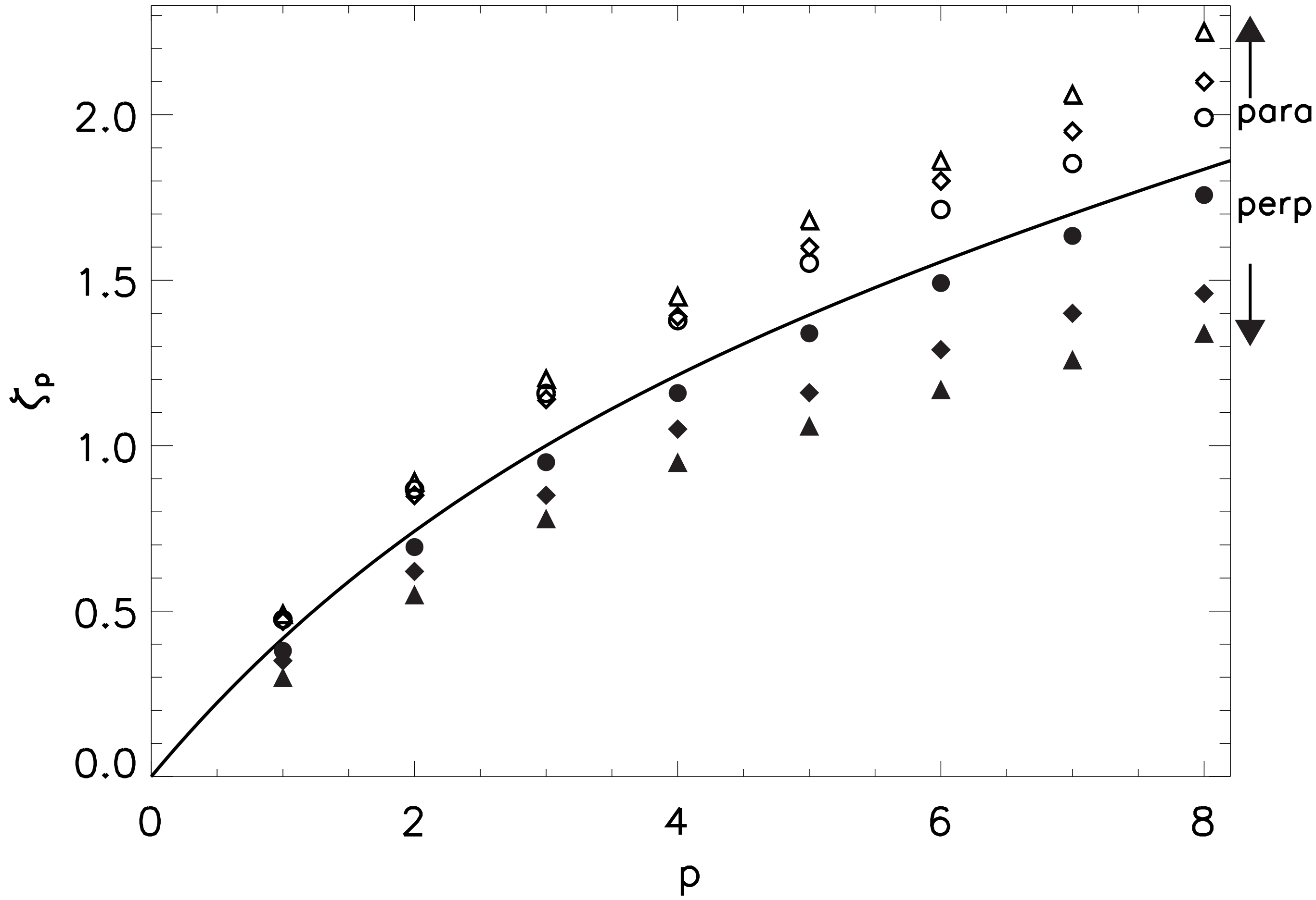
Dissipation by most intermittent structures $\varepsilon_\ell^{(\infty)} \sim \delta E^\infty / t_\ell^\infty$

- $t_\ell^\infty \sim \ell^x$, time-scale of most-singular dissipation.
- $v_\ell \sim \ell^{1/g}$, turbulent field scaling.
- $C_0 = x/(1-\beta)$, co-dimension of most singular structures.

$$\Rightarrow \zeta_p = \frac{p}{g}(1-x) + C_0 \left[1 - (1 - x/C_0)^{p/g}\right]$$

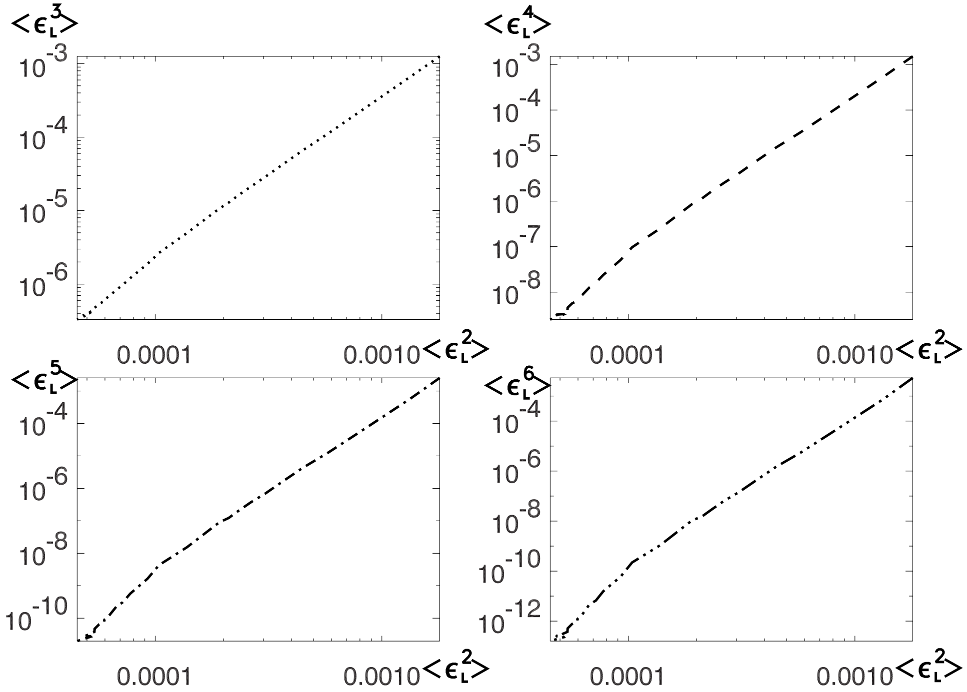
She & Lévêque PRL ’94, Grauer, Krug & Marliani Phys.Lett.A ’94, Politano & Pouquet PRE ’95

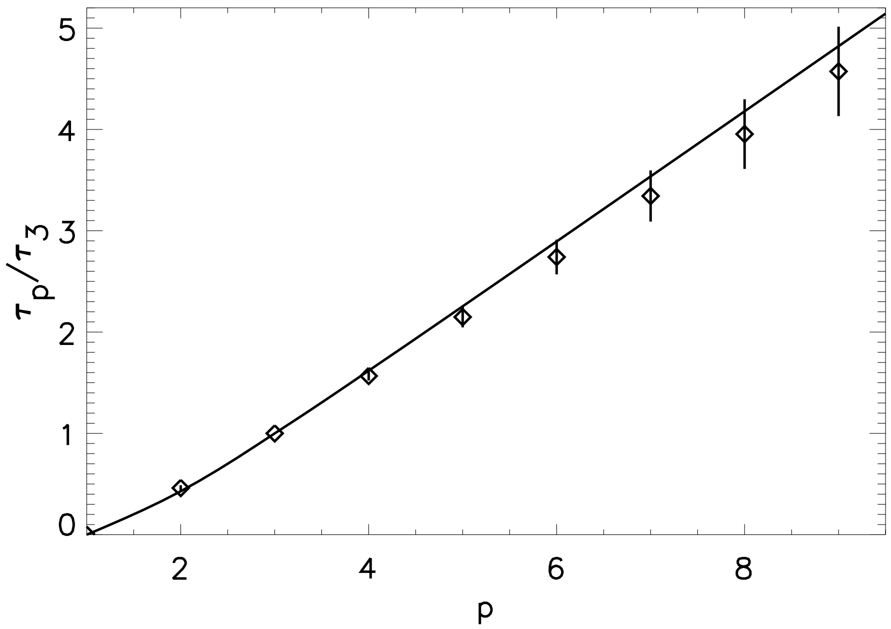
Anisotropic Two-Point Statistics



Filled symbolds: field perpendicular
Open symbols: field parallel

Refined Self-Similarity Hypothesis





Dissipation moments $\langle \varepsilon_\ell^p \rangle \sim \ell^{\tau_p}$ exhibit ESS

Log-Poisson model predicts (under assumption of refined self-similarity, $\zeta_p = p/g + \tau_{p/g}$)

$$\tau_p = -xp + C(1 - (1 - x/C)^p)$$

in accordance with simulations

Merrifield et al. Phys. Plasmas '05

Summary

- Isolated two numerical model systems for incompressible MHD turbulence
  - isotropic system: Kolmogorov cascading and excess magnetic energy at large scales
  - anisotropic system: Alfvénic/2D ($\perp \mathbf{B}_0$) and equipartition of kinetic/magnetic energy
- Kinetic/Magnetic energy spectra: equilibrium of small-scale dynamo$\longleftrightarrow$Alfvén effect
- Transition towards 2D (strong $\mathbf{B}_0$) detected and modelled via intermittency of dissipation
- Indication that anisotropy exhibits 'critical balance' scaling $k_{\|} \sim k_{\perp}^{2/3}$
- Refined similarity hypothesis for MHD turbulence verified