Turbulence Overshoot in Astrophysical Plasmas

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What does the term "overshoot" mean?

- Usually the term refers to *convective overshoot* – the incursion of motions driven by convection into stably stratified regions.
- More generally, it could refer to the spreading of turbulence from a region in which it is driven – e.g. by some hydrodynamic or MHD instability mechanism – into a region stable to such an instability.
- Overshooting versus penetration (often used interchangeably in astrophysics, but there is a well-defined distinction).

Why is overshoot important in astrophysics?

- In the long term it can have a significant effect on stellar evolution (timescales of $O(10^9)$ years).
- It can also play an important role in the short term dynamics of a star's magnetic field – e.g. seems to be a key component of the solar dynamo (timescales of $O(10)$ years).

Penetration vs. Overshoot

- In the absence of impermeable boundaries, flows resulting from convection will extend beyond the region of convective driving.

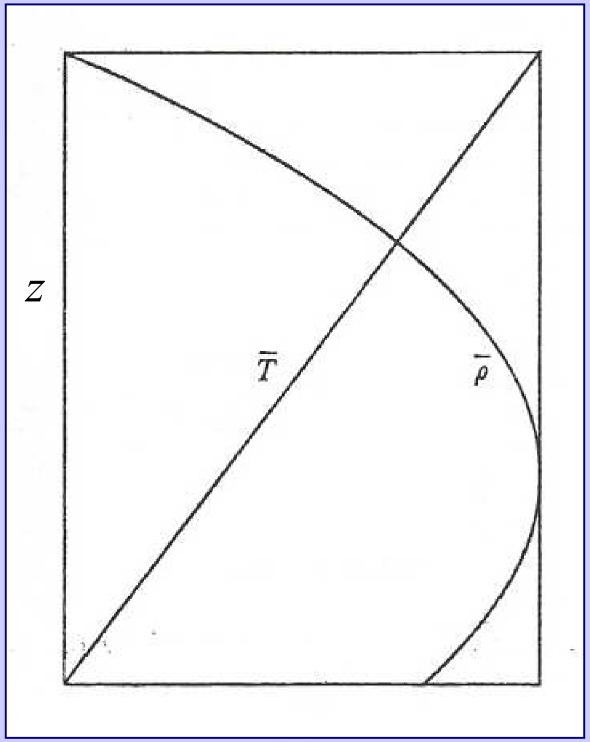
- If the resulting mixing is such as to alter the background state, thereby extending the region of instability, this is known as *penetrative convection.*

- If motions continue, through inertia, into surrounding stable regions (which maintain their stable stratification) then this is known as *overshooting convection.*

- In an experiment (real or numerical) it is possible to identify a region of penetrative convection. In a star this is less clear, since the initial background state is unknown.

Penetrative Convection: Simplest Model

Ice-water experiment: lower boundary at 0°C, upper boundary at $T > 4$°C.
(Malkus, Veronis, Moore & Weiss)



Basic state:

$$T = \lambda T_0 z, \quad \rho = \rho_0 (1 - \alpha (T - T_0)^2)$$

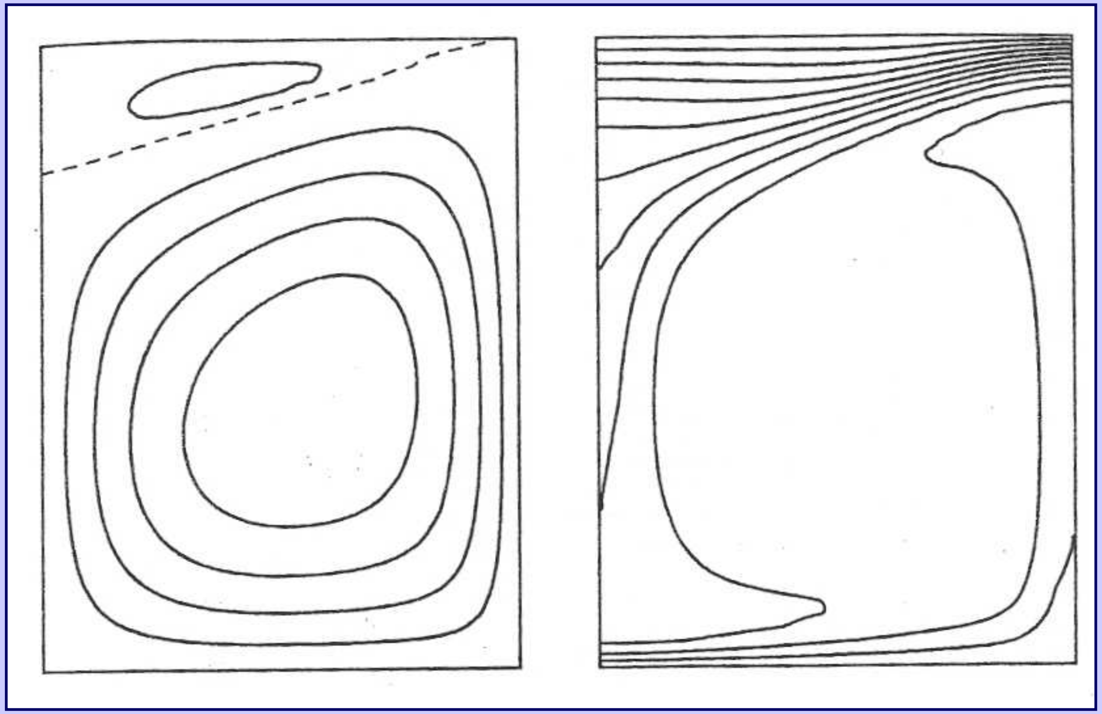
Key parameter: Rayleigh number

$$R = \frac{g d^3}{\kappa \nu} \left(\frac{\Delta \rho}{\rho_0} \right)$$

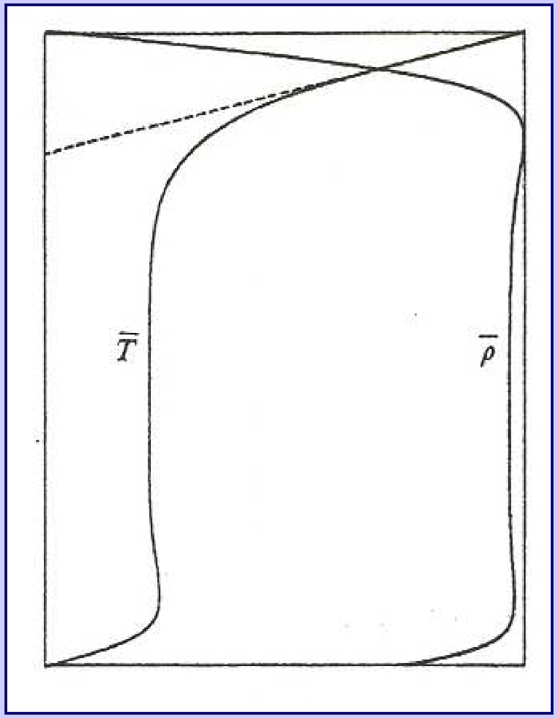
$\Delta \rho$ density change over unstable region

Penetrative Convection II

$R = 21 R_c$



Streamlines Isotherms



Mean profiles

(Moore & Weiss 1973 JFM)

Nonlinear modification to temperature and density profiles.

Theory of Stellar Structure

- One-dimensional theory. Gravity balanced by pressure.
- Magnetic fields, rotation ignored at this level.

(i) $$\frac{dP}{dr} = -\rho \frac{GM}{r^2}$$ (Hydrostatic equilibrium)

(ii) $$\frac{dM}{dr} = 4\pi r^2 \rho$$ (Conservation of mass)

(iii) $$\frac{dL}{dr} = 4\pi r^2 \rho \varepsilon$$ (Conservation of energy)

(iv) $$\frac{dT}{dr} = -\frac{3}{4ac}\frac{\kappa\rho}{T^3}\frac{L}{4\pi r^2}$$ (Radiative energy transport)

+ equation of state, equation for absorption coefficient, equation for energy generation

Stability of the Temperature Gradient

Simple "parcel" argument leads to the following (Schwarzschild) criterion for instability:

$$\frac{1}{\rho}\frac{d\rho}{dr} > \frac{1}{\gamma P}\frac{dP}{dr}$$

Alternatively this can be expressed as:

$$\frac{d\ln T}{dr} > \left.\frac{d\ln T}{dr}\right|_{adiabatic}$$

or

$$\frac{d\ln T}{d\ln P} > \left.\frac{d\ln T}{d\ln P}\right|_{adiabatic} = \frac{\gamma - 1}{\gamma}$$

Massive stars: convective cores
Less massive stars: convective envelopes

Mixing Length Theory (dimensionless analysis)

Hot bubbles rise, travel a distance ℓ adiabatically, and then "dissolve".
Cool elements sink and mix similarly.

Density deficit of rising element: $\Delta\rho \approx -\dfrac{\rho}{T}\Delta T$

Buoyancy force: $f = -g\Delta\rho = \dfrac{g\rho}{T}\Delta T$

Equation of motion: $\dfrac{d^2 r}{dt^2} = g\dfrac{\Delta T}{T}$ Hence $v \approx \left(g\ell\Delta T / T\right)^{1/2}$

Convective energy flux $\mathfrak{F}_{conv} = \dfrac{1}{2}\rho v c_p \Delta T = \dfrac{1}{2}\left(g\ell\Delta T / T\right)^{1/2}\rho c_p \Delta T$

or $\mathfrak{F}_{conv} = \dfrac{1}{2}\rho c_p \left(\dfrac{g}{T}\right)^{1/2} \ell^2 \left(\dfrac{d\Delta T}{dr}\right)^{3/2}$, where $\left(\dfrac{d\Delta T}{dr}\right)$ is superadiabatic gradient.

Mixing Length Theory II

What superadiabatic gradient is needed if all heat transport is by convection?

$$L(r) = 2\pi r^2 c_p \rho \left(\frac{g}{T}\right)^{1/2} \ell^2 \left(\frac{d\Delta T}{dr}\right)^{3/2}.$$

So, if we knew ℓ we could calculate $d\Delta T/dr$. But we don't.

If we write $\ell = \alpha R$ (R = solar radius) then, for the Sun:

$$\left(\frac{d\Delta T}{dr}\right) \approx \frac{10^{-18}}{\alpha} \deg/cm.$$

Actual temperature gradient ~ $T_{surf}/R \approx 10^{-4}$ deg/cm. Thus superadiabatic gradient very small. Good approximation to say that if convection takes place then temperature gradient is adiabatic.

Standard approach thus takes no account of convective overshooting.

Influence of Overshooting (Penetration) on Stellar Evolution

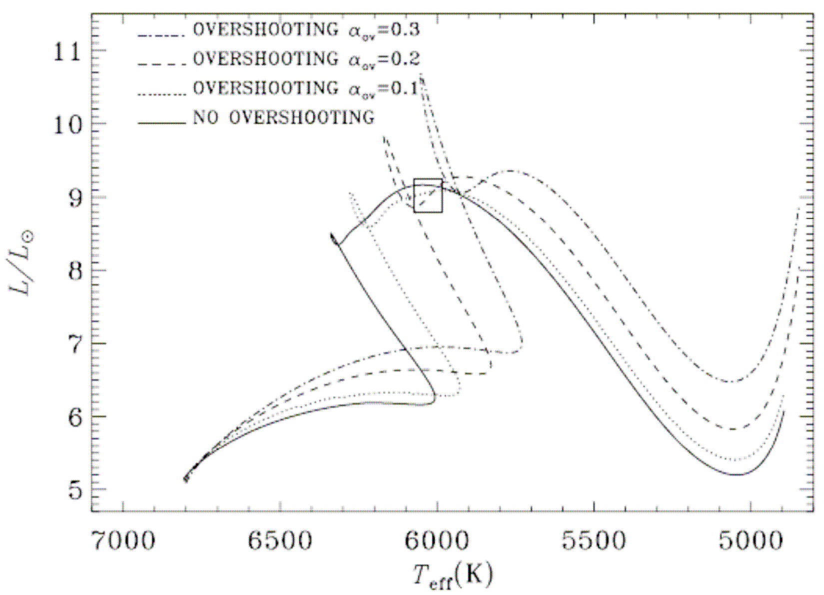
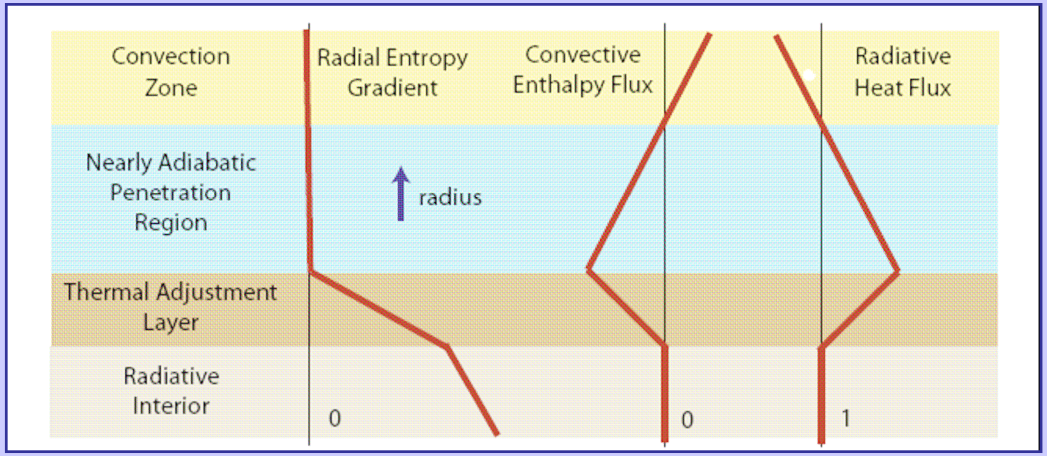


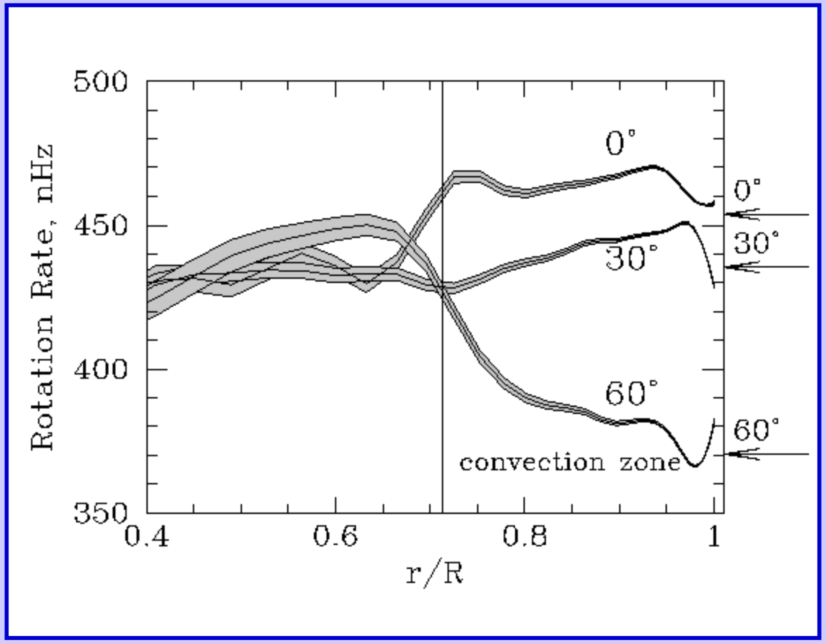
Fig. 3. The effect of the overshooting on the evolution of η Boo in the H–R diagram, calculated for increasing values of the extent of the overshooting from the the convective stellar core. Here models are calculated with $M = 1.7\ M_\odot$ and $Z = 0.04$.

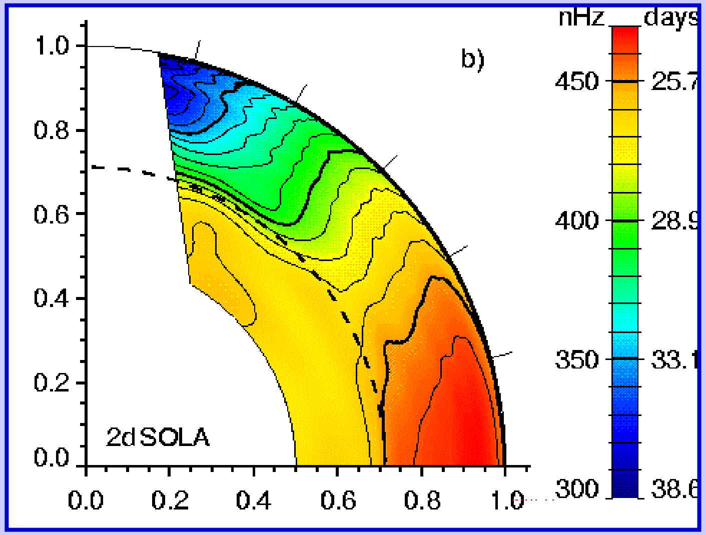
(Di Mauro *et al*. 2003 A&A)

Overshooting in the Sun: The Tachocline



(after Zahn 1991 A&A)





(Schou *et al*. 1998 ApJ)

Escape of Magnetic Field from the Tachocline

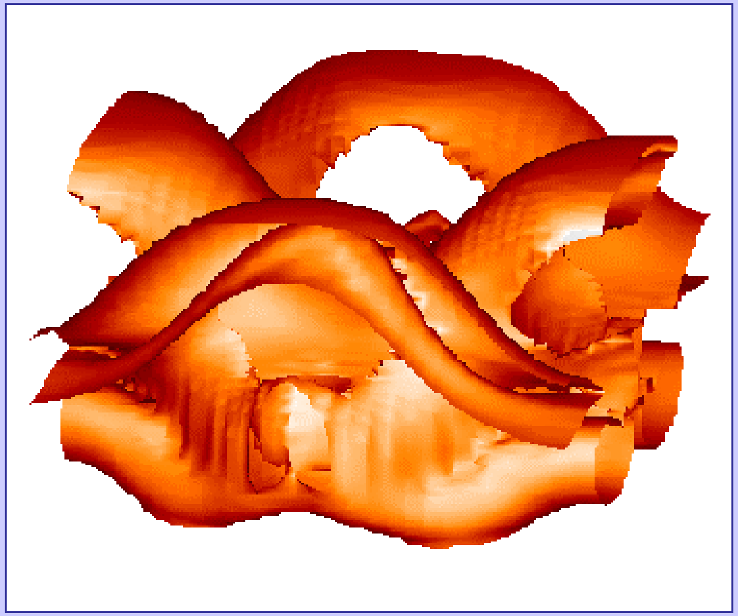
Important problem is one of magnetic confinement.
Crucial ingredient is stratification of background state, and also values of diffusivities.

Instability to magnetic buoyancy if:

$$-\frac{d}{dz}\ln B > \frac{\eta}{\kappa}N^2$$

(linear criterion)

Actually things are much more complicated since we need to understand the influence of velocity shear on magnetic buoyancy instabilities.



Matthews, DWH. & Proctor

Nonlinear evolution

Tackling Convective Overshoot

Theoretically

- Non-local MLT (Shaviv & Salpeter)
- Closure models (Canuto et al)
- Overshooting cellular convection (van Ballegooijen)
- Plume models of convection (Schmitt, Rosner & Bohn; Rempel)

Numerically

- Modal calculations (Latour *et al*., Toomre *et al*.)
- Fully compressible convection simulations:
  2-d (Hurlburt *et al*., Roxburgh & Simmons)
  3-d (Brummell *et al*., Singh *et al*., Nordlund *et al*.)
- Anelastic convection simulations:
  2-d (Rogers & Glatzmaier)

Non-local Mixing Length Theory

(Shaviv & Salpeter 1973 ApJ)

$$\Delta T(r_2, r_1) = -\int_{r_1}^{r_2} dr \left(\frac{dT}{dr} - \left.\frac{dT}{dr}\right|_{ad} \right)$$

$$v^2(r_2, r_1) = 2\int_{r_1}^{r_2} dr \frac{g(r)}{T(r)} \Delta T(r_2, r_1)$$

$$\mathfrak{F}_{conv}(r) = f c_p \rho(r) v(r, r-\ell) \Delta T(r, r-\ell)$$

$$\mathfrak{F}_{tot}(r) = \mathfrak{F}_{rad}(r) + \mathfrak{F}_{conv}(r)$$

Equations can be solved iteratively once the superadiabatic gradient is known over one mixing length.

Plume Model of Overshoot

(Schmitt *et al* 1984; Morton, Taylor & Turner 1956)

Assume convection dominated by downward plumes.
Integrate continuity equation, Navier-Stokes equation and entropy equation over a horizontal cross-section of a given plume.

$$\frac{d}{dz}\left(\pi\rho_0 u_z w^2\right) = 2\pi\alpha\rho_0 u_z w \qquad \text{Mass}$$

$$\frac{d}{dz}\left(\pi\rho_0 u_z^2 w^2\right) = -2g\lambda^2\pi w^2\Delta\rho \qquad \text{Momentum}$$

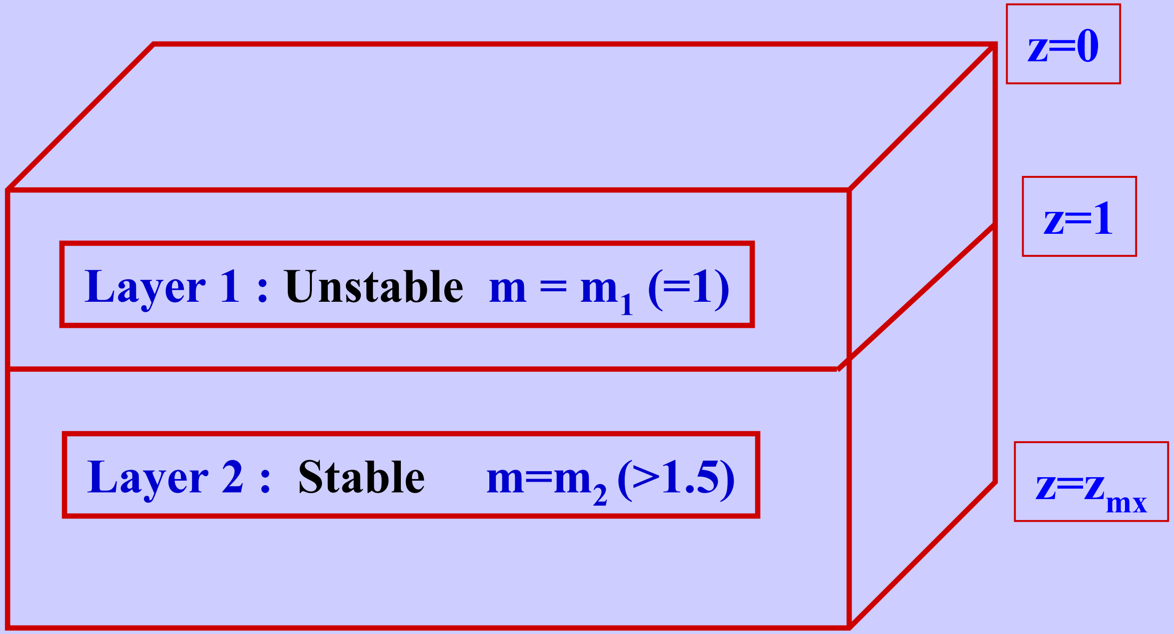
$$\frac{\lambda^2}{1+\lambda^2}\frac{d}{dz}\left(\pi\rho_0 u_z w^2\Delta S\right) = -\pi\rho_0 u_z w^2\frac{dS_0}{dz} + 2\pi\alpha\rho_0 u_z w\Delta S \qquad \text{Entropy}$$

α is an entrainment function, $\lambda = r_\rho / r_v$

Equations can be integrated in *z* from some initial condition.

Numerical Simulations of Convective Overshoot

- Compressible or Anelastic MHD
- DNS
- Cartesian geometry



Thermal diffusivity $\kappa=\kappa(z)$ (not $\kappa(\rho,T:x,y,z)$) : Piecewise polytropic atmosphere

$C_k(\text{layer1})/C_k(\text{layer2})=(m_2+1)/(m_1+1)$

"Stiffness", $S = (m_2-m_{ad})/(m_{ad}-m_1)$

High Ra, Pé, Low Pr

Governing Equations: Compressible Hydrodynamics

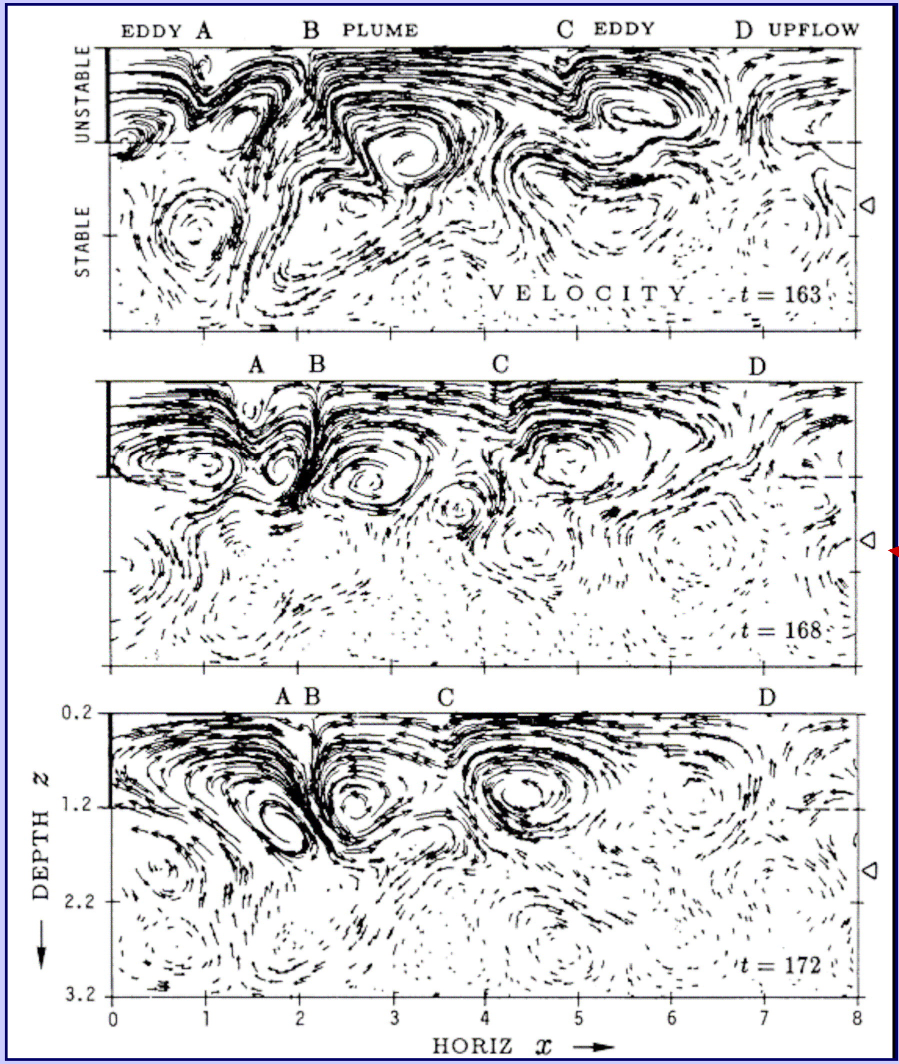
$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0,$$

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}\right) = -\nabla p + P_r C_k \left(\nabla^2 \mathbf{u} + \frac{1}{3}\nabla(\nabla \cdot \mathbf{u})\right) + \rho \mathbf{g},$$

$$\frac{\partial T}{\partial t} + \nabla \cdot (T\mathbf{u}) + (\gamma - 2) T \nabla \cdot \mathbf{u} = \frac{\gamma C_k}{\rho} \nabla \cdot (K_z \nabla T) + V_\mu,$$

$$p = \rho T.$$

Two Dimensional Simulations I



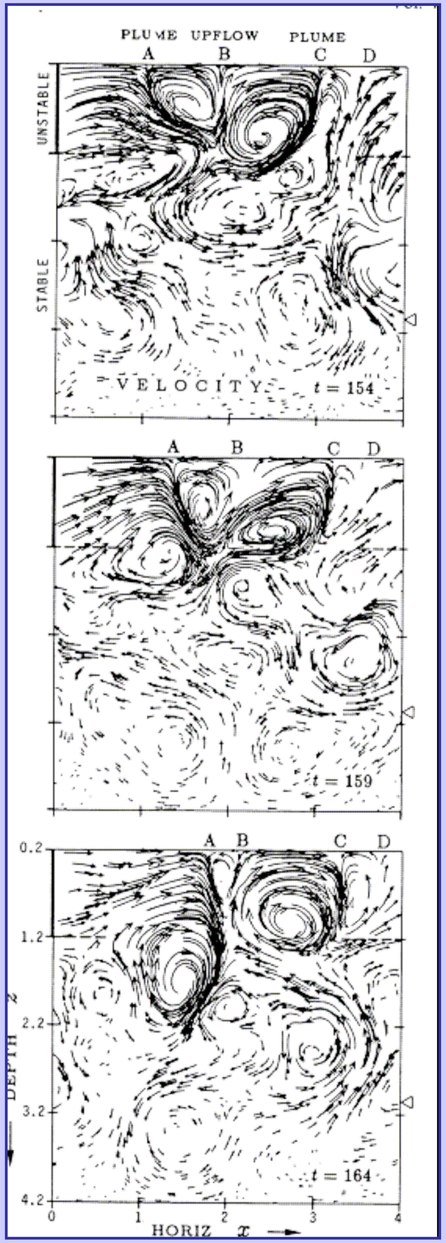
S = 3

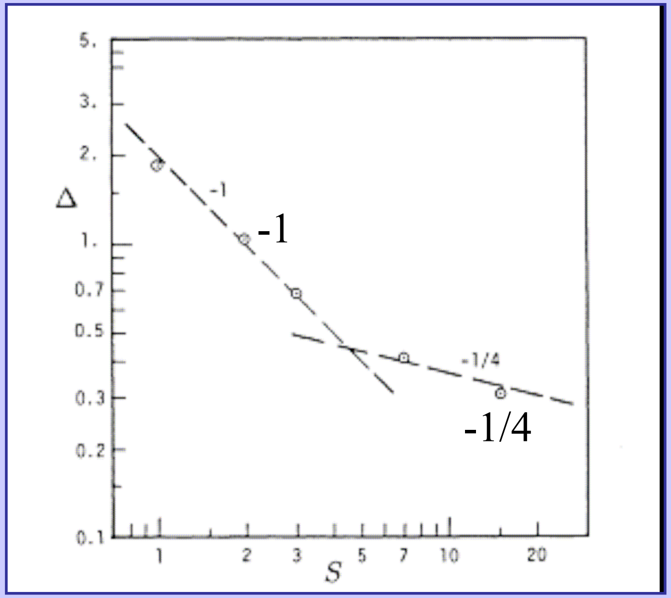
Base of penetration region

(Hurlburt *et al*. 1994 ApJ)

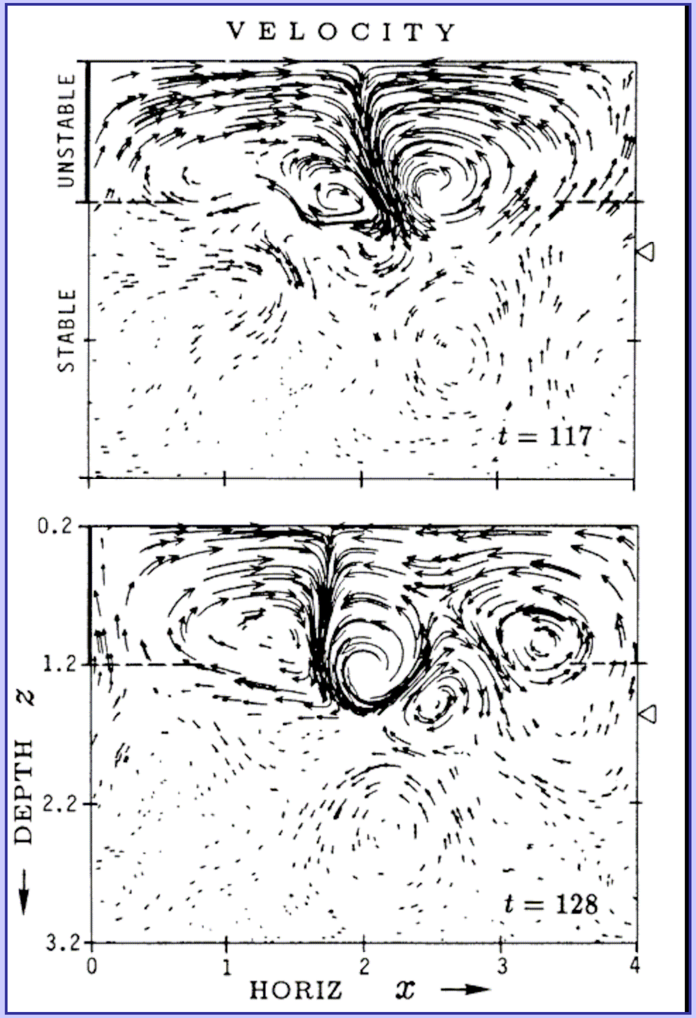
Two Dimensional Simulations II

$S = 1$





$S = 15$



Governing Equations: Anelastic Hydrodynamics

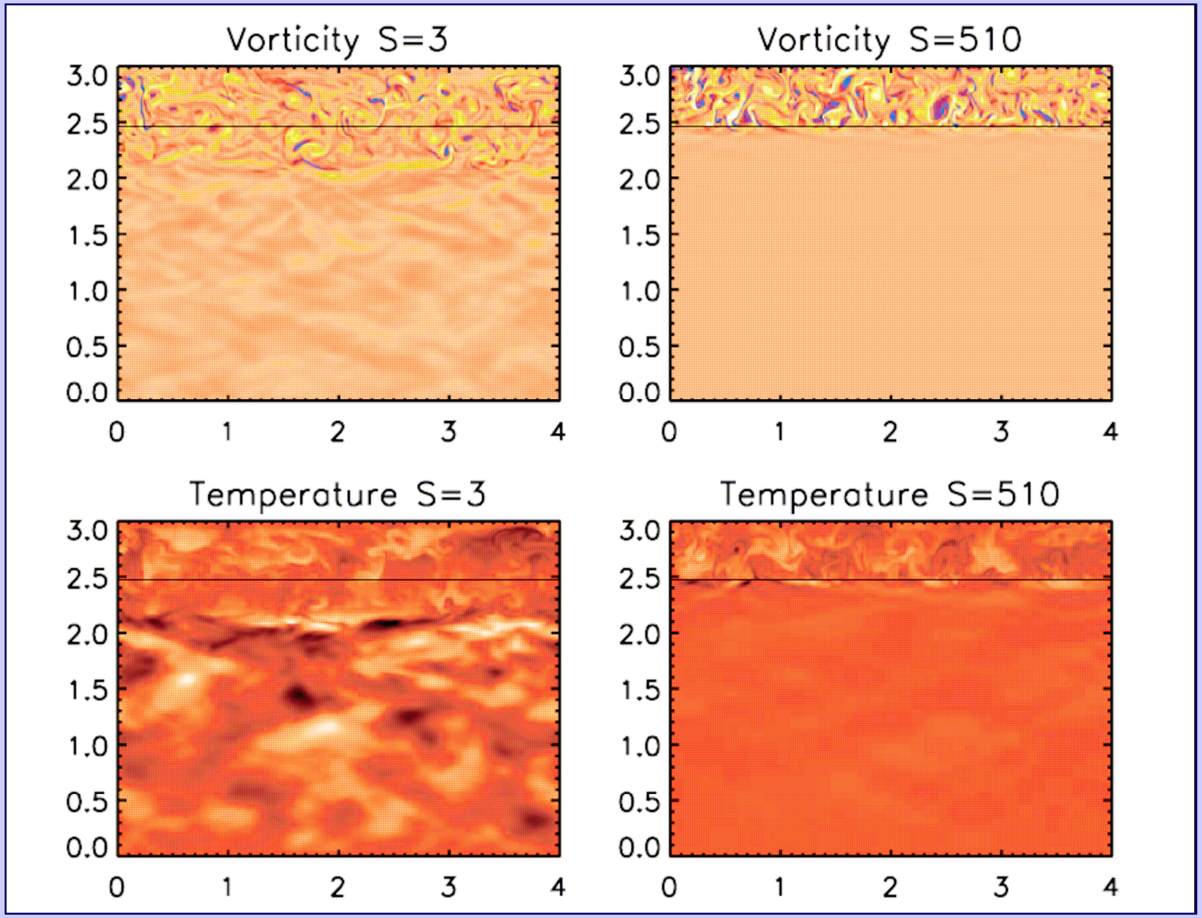
$$\nabla \cdot (\overline{\rho}\mathbf{u}) = 0,$$

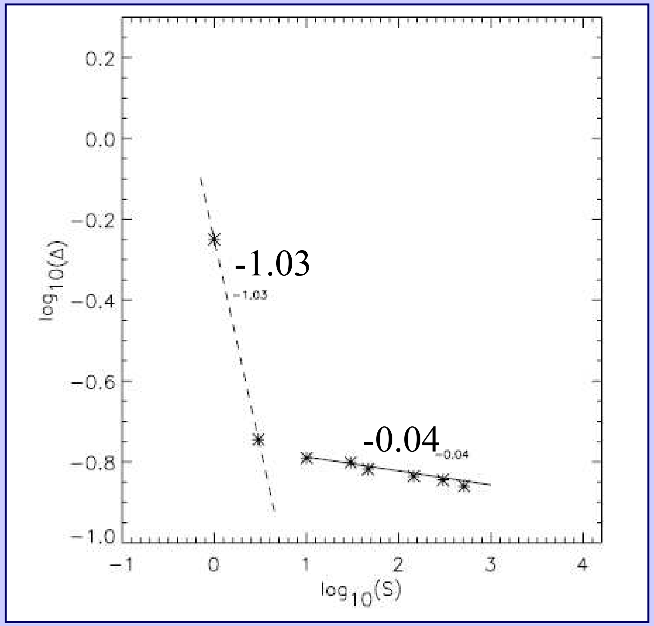
$$\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}\right) = -\nabla p + \overline{\nu}(\nabla^2 \mathbf{u} + \frac{1}{3}\nabla(\nabla \cdot \mathbf{u})) + C\mathbf{g},$$

$$\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = \text{usual stuff} + \overline{\mathrm{Q}},$$

where Q is the heating rate that maintains reference state temperature profile,
C **=** codensity perturbation.

Two Dimensional Simulations III





2D anelastic simulations (Rogers & Glatzmaier 2005 ApJ)

Three Dimensional Simulations I

Brummell, Cline & Toomre 2002 ApJ

High Peclet number

S=3

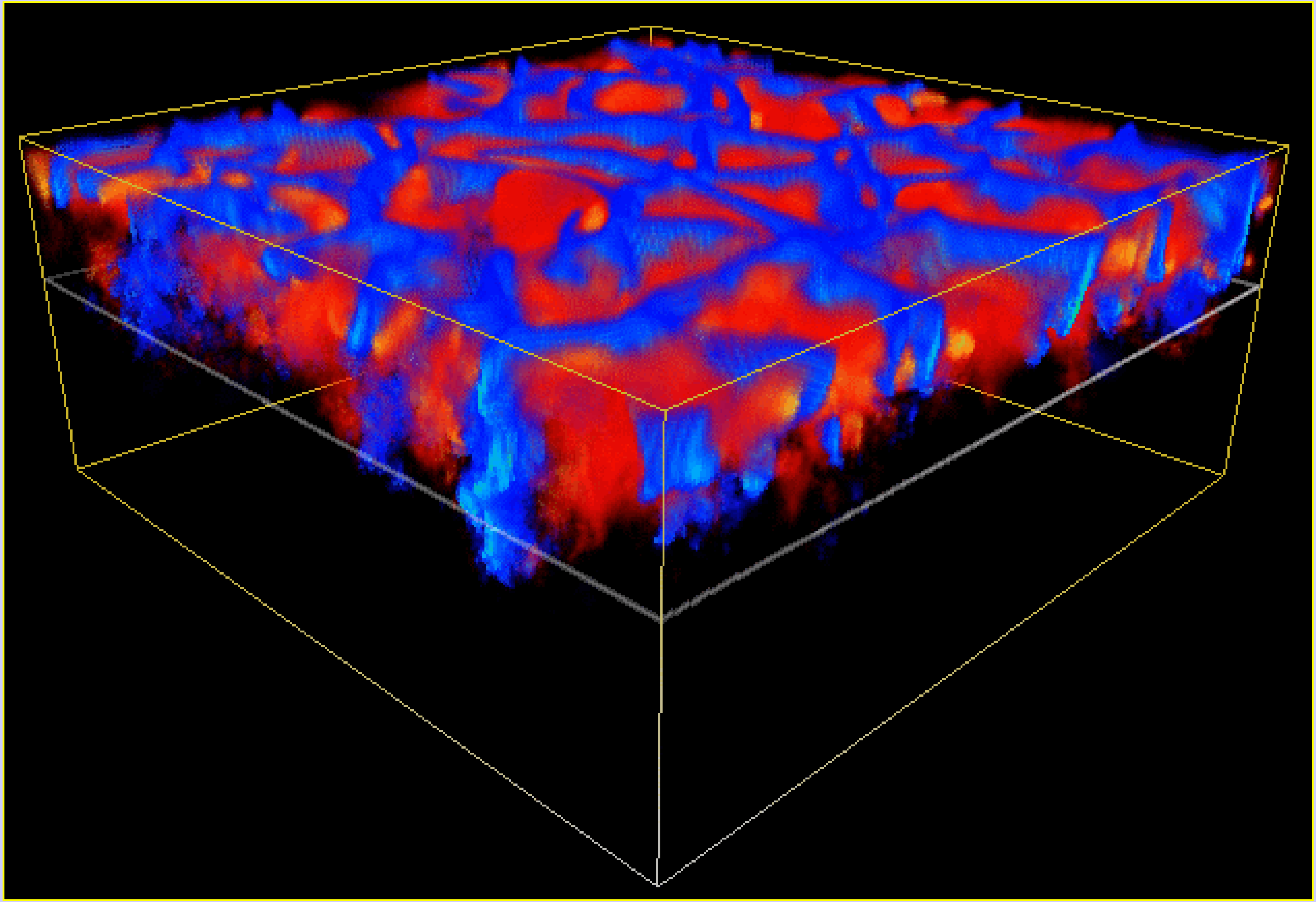
512x512x575

$Re_{rms} \sim 1800$

$Re_{\lambda} \sim 20$

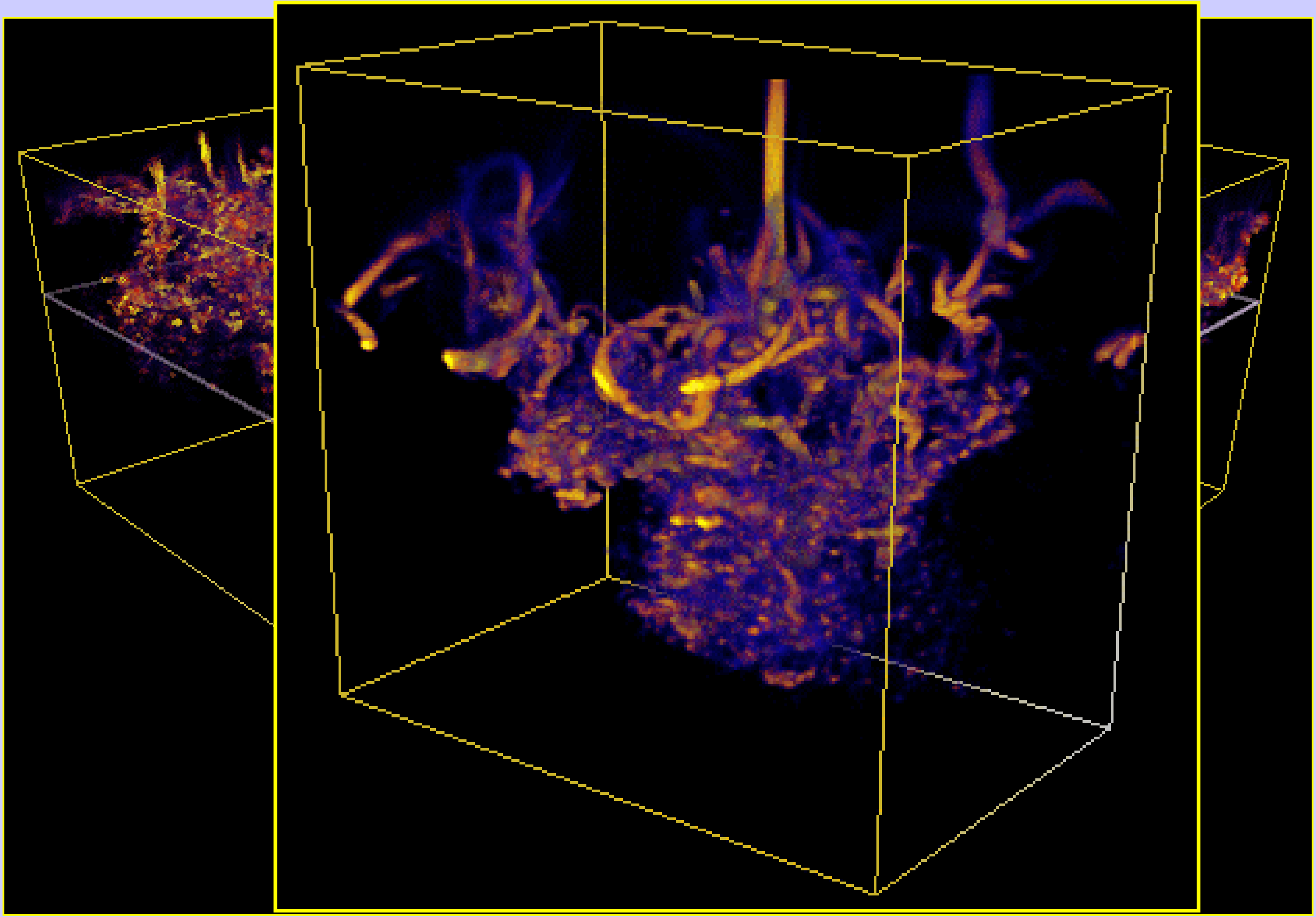
$Ra = 4x10^7$

$Pe_{down} \sim 200$



Vertical velocity, w

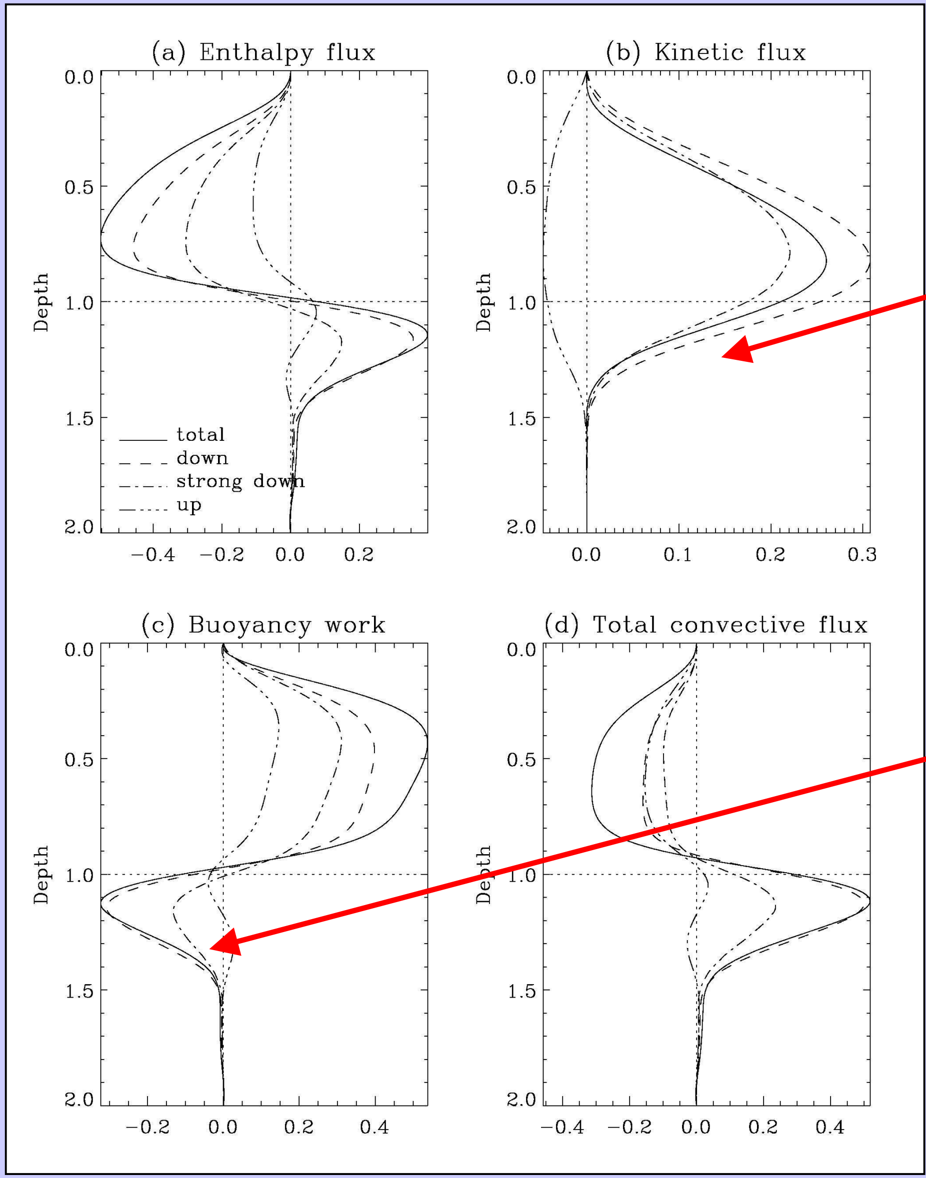
Three Dimensional Simulations II



Enstrophy density, ω^2

Brummell *et al* 2002

Three Dimensional Simulations III



¬ Overshooting motions extend below the interface.

¬ Large downwards (+ve) kinetic flux due to the strong downflows.

¬ Buoyancy braking decelerates the motions in the stable region.

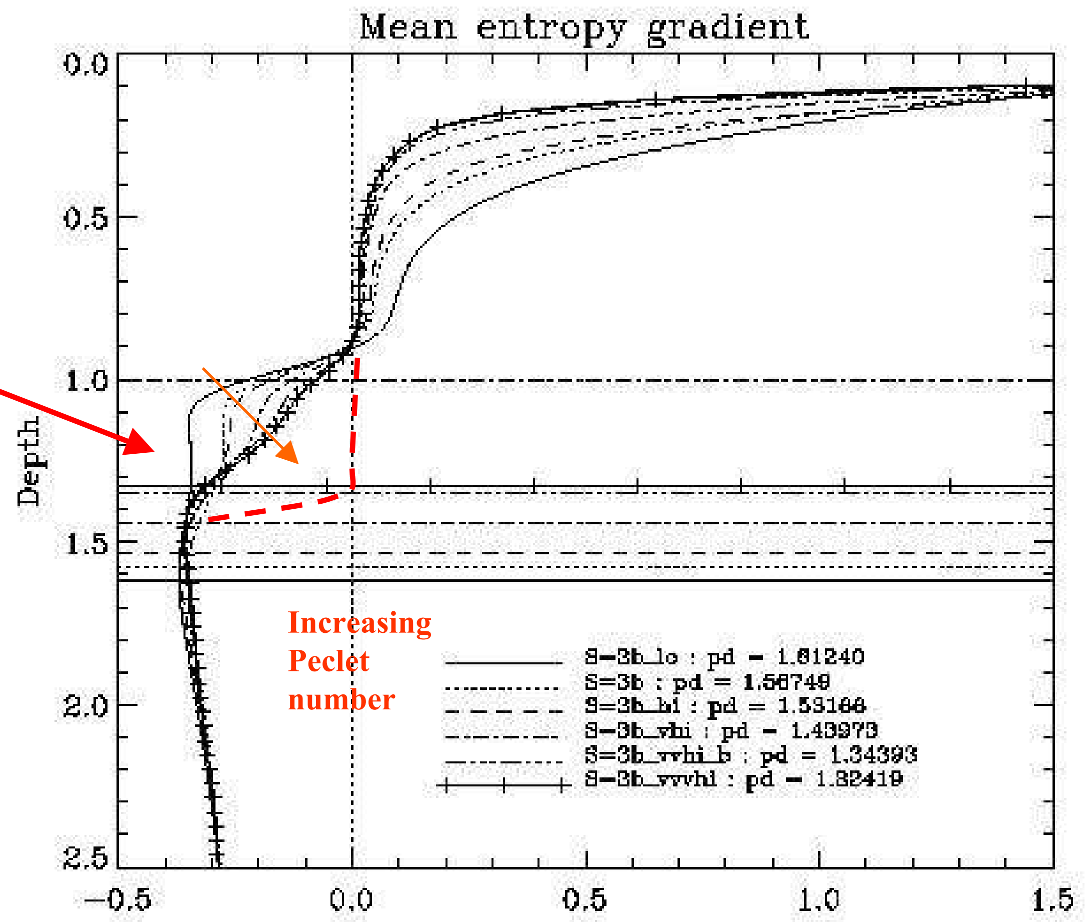
Brummell *et al* 2002

Three Dimensional Simulations IV

3-D penetrative convection does not really penetrate, only overshoots.

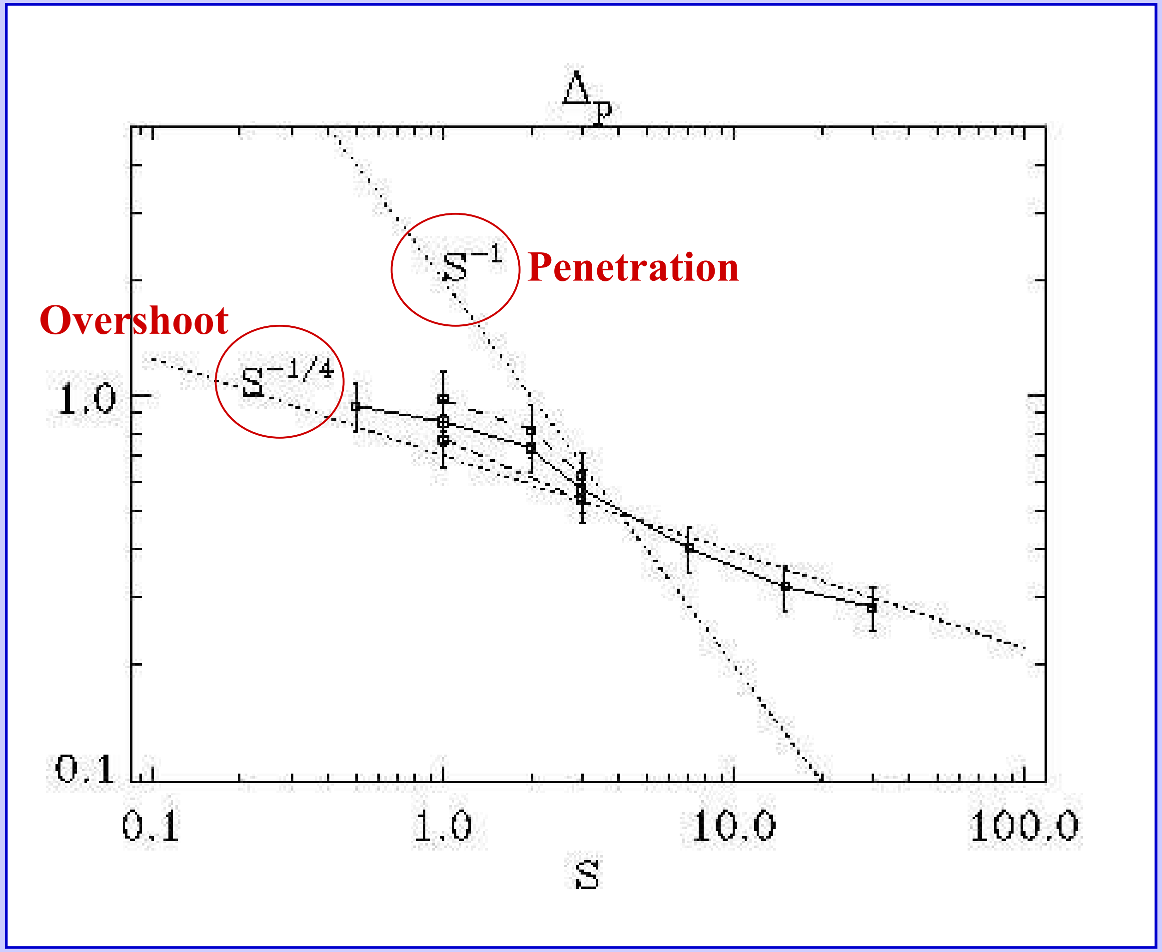
No extension of the adiabatically mixed region: due to low filling factor of 3-D plumes.

Even at highest Péclet numbers simulated.



Three Dimensional Simulations IV

¬ 3-D penetrative convection therefore has a different scaling with the relative stability of the lower layer than 2-D (Zahn, 1991), reflecting the lack of true penetration even at low S.



Transport: Mean Field Approach

How are scalars and vectors transported by turbulent flows?
Of particular significance: how are magnetic fields transported?

Two scale approach leads to mean field equations:

$$\frac{\partial \bar{\theta}}{\partial t} + \nabla \cdot \langle \mathbf{u}\theta' \rangle = \kappa \nabla^2 \bar{\theta}$$

$$\frac{\partial \bar{\mathbf{B}}}{\partial t} = \nabla \times \langle \mathbf{u} \times \mathbf{b} \rangle + \eta \nabla^2 \bar{\mathbf{B}}$$

And the ansatz:

$$F_i \equiv \langle u_i \theta' \rangle = -D_{ij} \frac{\partial \bar{\theta}}{\partial x_j} - E_{ijk} \frac{\partial^2 \bar{\theta}}{\partial x_j \partial x_k} + \cdots$$

$$\mathcal{E}_i \equiv \langle \mathbf{u} \times \mathbf{b} \rangle_i = \alpha_{ij} \bar{B}_j + \beta_{ijk} \frac{\partial \bar{B}_j}{\partial x_k} + \cdots$$

Mean Field Theory II

Symmetric part of α tensor leads to the famous α-*effect.*

Antisymmetric part gives:

$$\alpha_{ij}^{(a)} = \frac{1}{2}(\alpha_{ij} - \alpha_{ji}) = -\varepsilon_{ijk}\gamma_k \qquad \text{so} \qquad \alpha_{ij}^{(a)}\overline{B}_j = (\vec{\gamma} \times \vec{B})_i$$

γ (not necessarily solenoidal) can be non-zero if statistics of ***u*** are either anisotropic or inhomogeneous.

For the scalar problem, symmetric part of *D* leads to diffusion tensor.

Antisymmetric part gives:

$$\nabla \cdot \mathbf{F}^{(a)} = -\frac{\partial}{\partial x_i}\left(D_{ij}^{(a)} \frac{\partial \overline{\theta}}{\partial x_j} \right) = -\frac{\partial D_{ij}^{(a)}}{\partial x_i} \frac{\partial \overline{\theta}}{\partial x_j}$$

$$\nabla \cdot \mathbf{F}^{(a)} = \mathbf{V} \cdot \nabla \overline{\theta}, \qquad \text{where} \qquad V_j = -\frac{\partial D_{ij}^{(a)}}{\partial x_i}$$

V always solenoidal, regardless of the properties of ***u***. Need inhomogeneity for non-zero ***V*.**

(In 2-D, **γ** and ***V*** must be related – but they are not equal.)

Mean Field Theory III

Highly controversial issue concerns the suppression of the transport coefficients by the magnetic field.

For example, does α in the nonlinear regime take the form

$$\alpha = \frac{\alpha_0}{1 + B_0^2 / \langle u^2 \rangle} \qquad \text{or} \qquad \alpha = \frac{\alpha_0}{1 + RmB_0^2 / \langle u^2 \rangle} \qquad ?$$

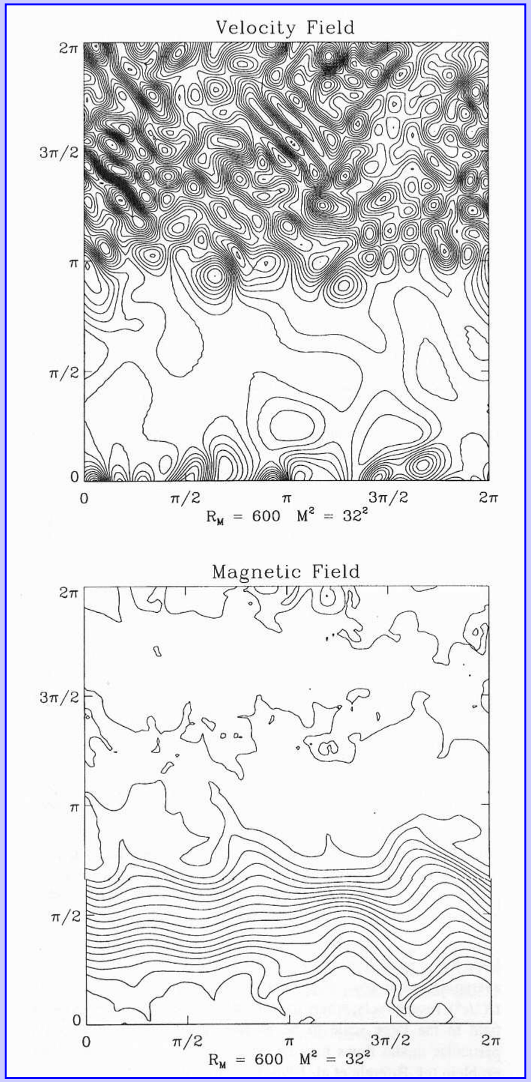
Less work on the suppression of γ.

Transport of Magnetic Flux: Computations

In 2-d, γ can arise only through inhomogeneity, not through anisotropy alone (cf. 3-d).

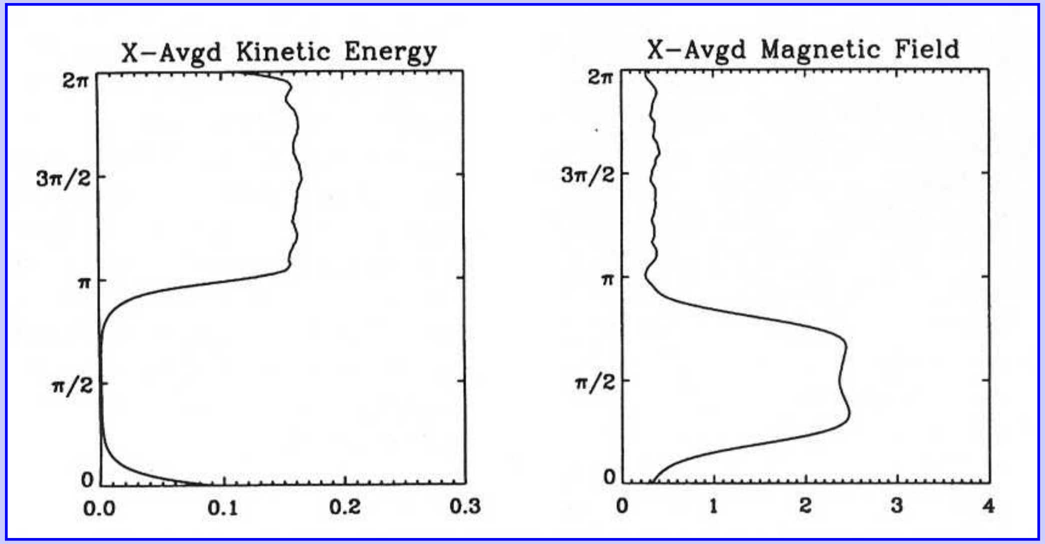
(Tao, Proctor & Weiss 1998 ApJ)

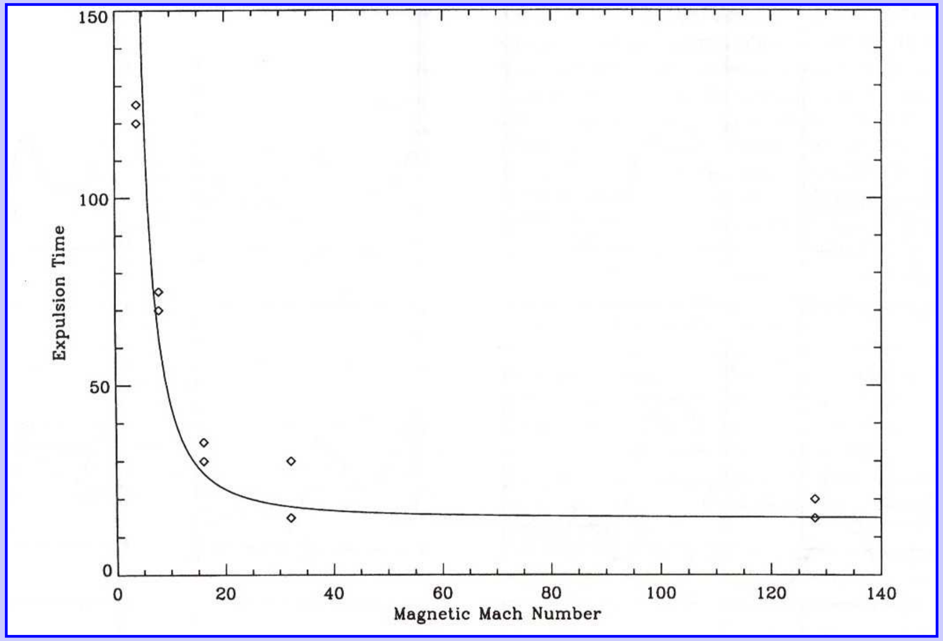
Periodic domain.
Uniform 'horizontal' magnetic field.
Turbulence confined to upper half of domain.



Transport of Magnetic Flux: Computations II

Time-averaged energy and field





Solid line:

$$\tau = 1500\left(\frac{1}{Rm} + \frac{1}{M^2}\right)$$

$M = U/V_A$ is magnetic Mach number

Conclusions

- Theories predict overshoot from a few to many percent of a pressure scale height.
- Simulations may be able to point the way to the correct scalings.
- How are magnetic fields, angular momentum transported by overshooting convection?
- Why is the tachocline so thin?
- How does the field subsequently escape?