Gyrokinetic Turbulence in Magnetized Plasmas

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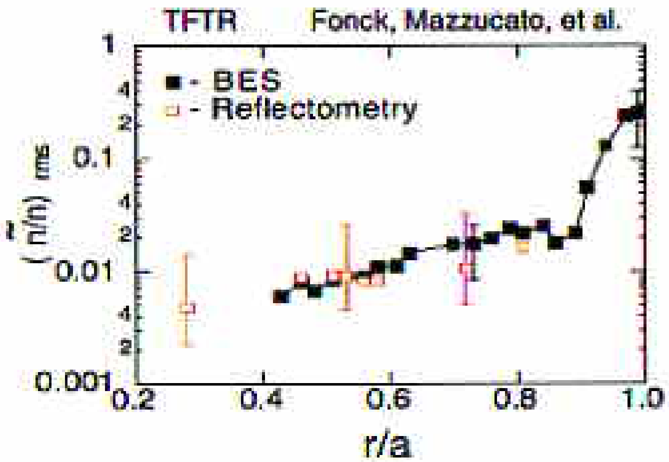
This Talk

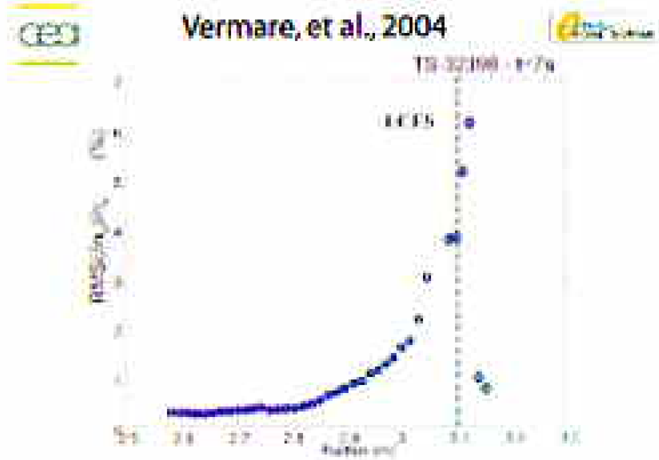
- Why people do gyrokinetics?
- **Modern Nonlinear Gyrokinetics:**
  - Emphasis on Conservation Laws
  - Systematic Derivation
  - Clear Pathways to Generalization/Extensions
- Focus on Tokamak Microturbulence
- Illustration of Prominent Examples

Outline

- Properties of Tokamak Micro-turbulence
- Standard Nonlinear Gyrokinetic Theory
- **Modern Nonlinear Gyrokinetics:**
  - Single Particle Dynamics and Gyrokinetic Vlasov Equation
  - Gyrokinetic Maxwell's Equation and Pullback Transformation
- Further Extensions

Amplitude of Tokamak Micro-turbulence





- Relative fluctuation amplitude $\delta n/n_0$ at core typically less than 1 %
- At the edge, it can be greater than 10 %

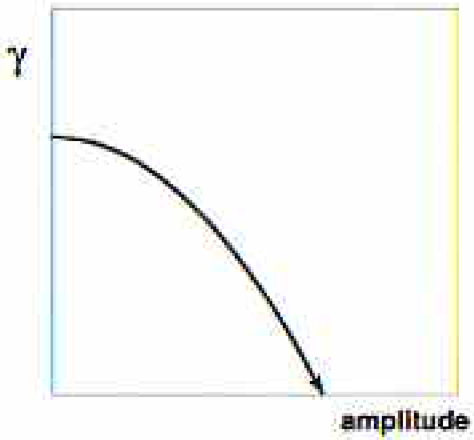
Properties of Tokamak Micro-turbulence

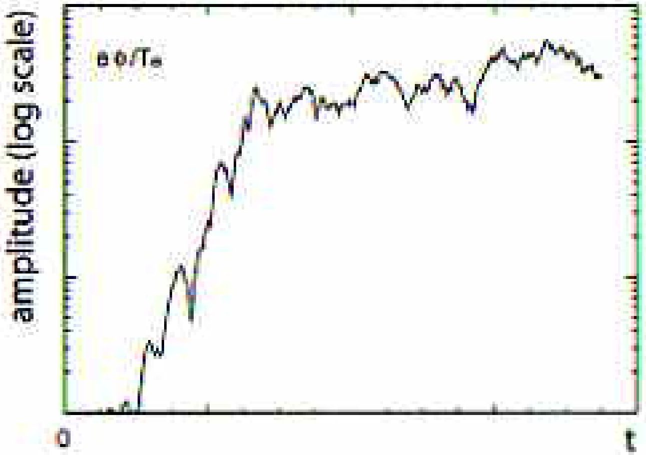
- $\delta n/n_0 \sim 1$ %
- $k_r \rho_i \sim k_\theta \rho_i \sim 0.1 \sim 0.2$
- $k_{||} < 1/qR << k_\perp$: Rarely measured
- $\omega - \mathbf{k} \cdot \mathbf{u_E} \sim \Delta\omega \sim \omega_{*pi}$:
  Broad-band, sometimes Doppler-shift dominates in rotating plasmas

Heuristic Estimation of Diffusion Coefficient

$$\gamma = \gamma_{lin} - k_\perp^2 D_{turb} \to 0$$

- Nonlinear coupling induced dissipation leads to saturation (B. Kadomtsev '65)





- $D_{turb} \sim \gamma_{lin}/k_\perp^2 \sim (v_{Ti}/a)\rho_i^2$: GyroBohm scaling
- "**Local** Balance in Space" for a mode **k**
- "Conceptual Foundation of Most Transport Models"
- **Missing:**
  - Nonlocal Phenomena: Turbulence Spreading,...

Standard Nonlinear Gyrokinetic Ordering

[Frieman and Chen, Phys. Fluids 1982]

Minimum number of ordering assumptions

- $\frac{\omega}{\Omega_i} \sim \frac{k_\parallel}{k_\perp} \sim \epsilon << 1$
- $k_\perp \rho_i \sim 1$ for generality:
  Short wavelength modes (with higher γ_{lin}) can affect the modes at NL peak ($k_\perp \rho_i \sim 0.1 \sim 0.2$)
  through NL coupling.
  $\rightarrow \omega \sim k_\parallel v_{Ti}$ for wave-particle resonance
  i.e., Landau damping
- $\frac{\delta f}{f_0} \sim \frac{e\delta\phi}{T_e} \sim \frac{1}{k_\perp L_p} \sim \epsilon << 1$
  - $k_\perp \frac{e\delta\phi}{T_e} \sim \frac{1}{L_p}$: $\mathbf{E} \times \mathbf{B}$ Nonlinearity $\sim$ Linear Drive
  - $\delta n / n_0 \sim \rho / L \sim$ roughly experimental vaues.

Conventional Nonlinear Gyrokinetic Equation

[eg., Frieman and Chen, Phys. Fluids 1982]

- Foundations of Tokamak Nonlinear Kinetic Theory
  for analytic applications, ballooning codes...
- Ordering is minimal and generic
- Based on direct gyro-phase average of Vlasov equation
  Lots of algebra and book keeping
- Direct expansions in ϵ: Self-consisten up to $O(\epsilon^2) \rightarrow$
  Should be fine for linear and nonlinear saturation phase
- Velocity space nonlinearity: $\nabla_{\|}\delta\phi\partial_{v_{\|}}\delta f \sim O(\epsilon^3)$ is ignored.
  Energy, phase space volume **not** conserved.
- May not be able to describe long term behavior accurately
  [Villard, Hatzky, Sorge, Lee, Wang]
  $\rightarrow$ Physics responsible for difference?

Modern Nonlinear Gyrokinetics

- Starting from the original Vlasov-Maxwell system (6D), pursue **"Reduction of dimensionality"** for both computational and analytic (cf. MNR) feasibility.
- Keep intact the underlying symmetry/conservation of the original system.
- Perturbation analysis consists of near-identity coordinate transformation which **"decouples"** the gyration from the slower dynamics of interest in the single particle Lagrangian, rather than a direct "gyro-phase average" of Vlasov equation.

Phase Space Lagrangian Derivation of Nonlinear Gyrokinetics

[since Hahm, PF **31**, 2670 '88, followed by Brizard, Sugama,...]

- **Conservations Laws are Satisfied.**
- Various expansion parameters appear at different stages $\rightarrow$ Flexibility in variations of ordering for specific application
- Guiding center drift calculations in equilibrium field $\mathbf{B}$: Expansion in $\delta_B \equiv \rho_i / L_B \sim \rho_i / R$.
- Perturbative analysis consists of near-identity transformations to new variables which remove the gyro-phase dependence in perturbed fields $\delta\mathbf{A}(\mathbf{x}), \delta\phi(\mathbf{x})$ where $\mathbf{x} = \mathbf{R} + \rho$: Expansion in $\epsilon_\phi \equiv e(\delta\phi - \frac{v_\parallel}{c}\delta A_\parallel)/T_e \sim \delta B_\parallel / B_0$
- Derivation more transparent, less amount of algebra

Single Particle Phase Space Lagrangian

[Littlejohn, Cary '83,...]

- Fundamental 1-form (phase space Lagrangian in non-canonical variables)

$$\gamma \equiv (e\mathbf{A}(\mathbf{x}) + m\mathbf{v}) \cdot d\mathbf{x} - (m/2)v^2 dt$$

- Transformation to guiding center variables: $\mathbf{x} \equiv \mathbf{R} + \rho$, $\mu \equiv v_\perp^2/2\Omega$, $\theta \equiv tan^{-1}(\frac{\mathbf{v}\cdot\mathbf{e}_1}{\mathbf{v}\cdot\mathbf{e}_2}), \ldots$
- The zero-th order phase space Lagrangian for guiding center:

$$\gamma_0 = (e\mathbf{A} + mv_{\|}\mathbf{b}) \cdot d\mathbf{R} + \frac{\mu B}{\Omega} d\theta - H_0 dt$$

angle variable θ is ignorable
action is an adiabatic invariant μ

$$H_0 = \mu B + (m/2)v_{\|}^2$$

Euler-Lagrange Equation

- From variation of phase space Lagrangian:

$$\frac{d\theta}{dt} = \Omega, \frac{d\mu}{dt} = 0$$

Decoupling of gyromotion, adiabatic invariant

- $$-e\mathbf{B}^* \times \frac{d\mathbf{R}}{dt} - m\mathbf{b}\frac{dv_\parallel}{dt} = \mu \nabla B$$

where $\mathbf{B}^* \equiv \nabla \times (\mathbf{A} + \frac{m}{e} v_\parallel \mathbf{b}) = \mathbf{B} + \frac{m}{e} v_\parallel \nabla \times \mathbf{b}$

- Decompose via $\mathbf{b}\times$ and $\mathbf{B}^*$, to get

$$\frac{d\mathbf{R}}{dt} = v_\parallel \frac{\mathbf{B}^*}{B^*} + \frac{\mu}{e}\frac{\mathbf{b}}{B^*} \times \nabla B, \frac{dv_\parallel}{dt} = -\frac{\mu}{m}\frac{\mathbf{B}^*}{B^*} \cdot \nabla B$$

More on Guiding Center Drift

- Frequently asked question:
  "Where is the curvature drift?"
  Using an identity $\mathbf{B}^* = B^*\mathbf{b} + \frac{m}{e}v_\parallel \mathbf{b}\times(\mathbf{b}\cdot\nabla)\mathbf{b}$:

$$\frac{d\mathbf{R}}{dt} = v_\parallel \frac{B^*\mathbf{b} + \frac{m}{e}v_\parallel \mathbf{b}\times(\mathbf{b}\cdot\nabla)\mathbf{b}}{B^*} + \frac{\mu}{e}\frac{\mathbf{b}}{B^*}\times\nabla B$$

- Infrequently asked question: **"Do conventional guiding center drifts conserve energy?"**

$$\frac{d\mathbf{R}}{dt} = v_\parallel \mathbf{b} + \mathbf{v}_{curv} + \mathbf{v}_{gradB};\ \frac{dv_\parallel}{dt} = -\frac{\mu}{m}\mathbf{b}\cdot\nabla B$$

  do *not* conserve energy exactly, while our E-L eqns do.
- $\mathbf{B}^*$ is a manifestation of Hamiltonian structure
- B^* is the density of phase-volume, $d^6\mathbf{Z} = B^* d\mu d\theta dv_\parallel d^3\mathbf{R}$

Lie Perturbative Analysis

[from Hahm, PF **31**, 2670 '88]

- Consider electrostatic fluctuation only (for illustration): $\delta\phi(\mathbf{x}) = \delta\phi(\mathbf{R} + \rho)$
- While gyromotion has been decouple in the zero-th order phase space Lagrangian, it appears again in the perturbation. Since it is $O(\epsilon)$, we can remove it via *near-identity, phase-space preserving* Lie transform.
- In addition to zero-th order γ_0, $\gamma_1 = -e\delta\phi(\mathbf{R} + \rho)dt$
- Perform Lie-perturbation:

$$\Gamma_1 = \gamma_1 - L_1\gamma_0 + dS_1$$

where $(L_1\gamma)_\mu = g_1^\nu(\frac{\partial\gamma_\mu}{\partial z^\nu} - \frac{\partial\gamma_\nu}{\partial z^\mu})$, transformation of 1 form

Lie Perturbative Analysis

- One can choose the gauge function S_1 and the *generator* g_1 such that the gyrophase is removed from Γ_1
- We obtain,

$$\Gamma_1 = -e < \delta\phi > dt$$

where $< ... >$ is the gyrophase average $\frac{1}{2\pi} \int (...)$

- Now, $\Gamma = \Gamma_0 - e < \delta\phi > dt$,
$H = H_0 + H_1 = \mu B + (m/2)v_{\|}^2 + e < \delta\phi >$
- Note that *decoupled gyrophase information* is kept in $dS_1 = \frac{e}{\Omega}(\delta\phi - < \delta\phi >)d\theta$ and g_1 to be used later when necessary.
- The second order perturbation in $\epsilon_\phi \sim \rho/L_p$ is necessary for energy conservation.

Gyrokinetic Vlasov-Poisson System

- With Euler-Lagrange Eqns, Gyrokinetic Vlasov equation for gyrocenter distribution function $F(\overline{R}, \overline{\mu}, \overline{v}_{\|}, t)$ is:

$$\frac{\partial F}{\partial t} + \frac{d\overline{R}}{dt} \cdot \overline{\nabla} F + \frac{d\overline{v}_{\|}}{dt} \frac{\partial F}{\partial \overline{v}_{\|}} = 0$$

  Note reduction of dimensionality achieved by $\frac{\partial F}{\partial \theta} = 0, \frac{d\overline{\mu}}{dt} = 0$
- Self-consistency is enforced by the Poisson's equation. Debye shielding is typically irrelevant, one must express the ion particle density $n_i(\mathbf{x})$ in terms of the gyrocenter distribution function $F(\overline{R}, \overline{\mu}, \overline{v}_{\|}, t)$
- Lee [PF **26**, 556 '83] has identified the *polarization density* (in addition to the guiding center density). It was a key breakthrough in advances in GK particle simulations.

$$\delta n_i(\mathbf{x}) = \delta n_{gc} + \rho_i^2 \nabla_\perp \cdot N_0 \nabla_\perp (e\delta\phi / T_i)$$

Extensions to Edge

[for core transport barriers → Hahm, Phys. Plasmas **3**, 4658, '96]

Expansion in $\epsilon_B \sim \rho_i / L_E \sim \frac{B_\theta}{B}$:

- From $\rho_{ip} \sim L_P \sim L_E$,
  $u_E \sim u_{*i} \sim \frac{\rho_i}{L_p} v_{Ti}$, $\frac{e\Phi^{(0)}}{T_e} \sim 1$.
- $|S-1| \sim 1$ (banana orbit distortion), $\frac{\omega_E}{\Omega_i} \sim \epsilon_B^2$ (circular gyro-orbit)
  where $\omega_E \equiv \frac{(RB_\theta)^2}{B} \frac{\partial}{\partial \psi} \left(\frac{E_r}{RB_\theta} \right)$ [Hahm-Burrell, PoP '95]
  $S \simeq 1 + \left(\frac{B}{B_\theta} \right)^2 \frac{\omega_E}{\Omega_i}$ [Hinton-Kim, Furth-Rosenbluth, Shaing,...]

- The zero-th order phase space Lagrangian

$$\gamma_0 \equiv (e\mathbf{A} + m\mathbf{u}_E + mv_\parallel \mathbf{b}) \cdot d\mathbf{R} + \frac{\mu B}{\Omega} d\theta - H_0 dt$$

  with a guiding-center Hamiltonian

$$H_0 = e\Phi + \mu B + (m/2)(v_\parallel^2 + u_E^2) + \frac{\mu B}{2\Omega} \mathbf{b} \cdot \nabla \times \mathbf{u}_E.$$

Pullback Transformation

- More systematic derivation of GK Poisson's eqn started since Dubin *et al.*,[PF **26**, 3524 '83] via *pullback* transformation:

$$\nabla^2 \delta\phi = -4\pi e[\int d^6\overline{Z}\ (T_G^* \delta f)\ \delta^3(\overline{R} - \mathbf{x} + \overline{\rho}) - \delta n_e(\mathbf{x}, t)],$$

  where

$$T_G^* \delta f \equiv \delta f + \left(\frac{\partial S_1}{\partial \overline{\theta}}\right) \frac{\partial F_0}{\partial \overline{\mu}} + [\frac{1}{\Omega}\left(\overline{\nabla} S_1\right) \times \mathbf{b}] \cdot \overline{\nabla} F_0$$

- Contribution to the ion particle density which involves S_1 is the general form of polarization density. After linearization,

$$\{k^2 \lambda_{Di}^2\} \frac{e\delta\phi_{\mathbf{k}}}{T_{i\perp}} n_0 + \{1 - \Gamma_0(b)\} \frac{e\delta\phi_{\mathbf{k}}}{T_{i\perp}} n_0 = \delta\overline{N}_{i\mathbf{k}} - \delta n_{e\mathbf{k}}$$

- It is well known that the *polarization density* statisfies

$$\frac{\partial}{\partial t} \delta n^{pol} + \frac{\partial}{\partial \mathbf{x}} \cdot n_0 \mathbf{v}^{pol} = 0$$

  [eg., Fong and Hahm, PoP **6**, 188 '99]

Conservation of Energy and Phase-Space Volume

- It is straight-forward to show the Liouville's theorem:

$$\overline{\nabla}\cdot\left(B_{\|}^{*}\frac{d\overline{R}}{dt}\right)+\frac{\partial}{\partial\overline{v}_{\|}}\left(B_{\|}^{*}\frac{d\overline{v}_{\|}}{dt}\right)=0$$

- The invariant energy for GK Vlasov-Poisson system is obtained by transforming the energy constant of the original Vlasov-Poisson system [Dubin *et al.*,'83]

$$E=\int d^6\mathbf{Z}\,F_i(\mu B+\frac{M}{2}v_{\|}^2)+\int d^6\mathbf{z}\,f_e(\mathbf{z})\frac{1}{2}m_e v^2$$

$$+\frac{1}{8\pi}\int d^3\mathbf{x}\,|\mathbf{E}|^2+\frac{e^2}{2\Omega}\int d^6\mathbf{Z}\,F_i\left(\frac{\partial}{\partial\mu}\langle\delta\tilde{\phi}^2\rangle+\frac{1}{\Omega}\langle\nabla\delta\tilde{\Phi}\times\mathbf{b}\cdot\nabla\delta\tilde{\phi}\rangle\right)$$

Note that the last term can be obtained from perturbation up to $O(\epsilon_\phi^2)$.

Summary

- Modern Nonlinear Gyrokinetic Theory has provided a firm theoretical foundation for recent remarkable advances in gyrokinetic simulations and associated theories.
- Its elegance and relative simplicity have contributed to deeper understanding of the gyrokinetic system and its relation to other reduced system of equations.
- It should be useful for even more complicated systems where several expansion parameters exist.

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References on Nonlinear Gyrokinetic Theory I.

- Pioneering papers on conventional NL GK and NL GK for particle simulation:

  Frieman and Chen, PF **25**, 502 '82

  Lee, PF **26**, 556 '83
- Early Modern NL GK using Hamiltonian method (Darboux Theorem) in slab:

  Dubin, Krommes, Oberman, and Lee, PF **26**, 3524 '83 (Electrostatic)

  Hahm, Lee, and Brizard, PF **31**, 1940 '88 (Electromagnetic, canonical momentum formulation)
- Modern NL GK using phase-space Lagrangian Lie perturbation method:

  Hahm, PF **31**, 2670 '88 (General geometry, electrostatic)

  Brizard, J. Plasma Phys. **41**, 541 '89 (General geometry, electromagnetic)

References on Nonlinear Gyrokinetic Theory II.

- Robustness of NL GK formulation in the high amplitude DK regime:

  Dimits, Lodestro, Dubin, PF-B **4**, 274 (form of eqns unchanged from Hahm-Lee-Brizard '88)
- NL GK for strongly rotating plasmas:

  Brizard, PoP **2**, 459 '95 (in terms of toroidal rotation)

  Hahm, PoP **3**, 4658 '96 (in terms of E_r)
- Energy conservation theorem:

  Brizard, PoP **7**, 4816 '00

  Sugama, PoP **7**, 466 '00 (introduction of field theory)

References on Topics related to Modern NL GK using phase-space Lagrangian Method

- Bounce-averaged Nonlinear Kinetic equation

  Fong and Hahm, PoP **6**, 188 '99 (electrostatic)

  Brizard, PoP **7**, 3238 '00 (electromagnetic)

- High frequency linear gyrokinetic theory:

  Qin and Tang, PoP **11**, 1052 '04 (recovery of compressional Alfven wave, elucidation of differential geometrical meaning of pullback transformation