MHD Turbulence

An overview of theoretical uncertainties

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with thanks to my collaborators:
S. Cowley *(UCLA & Imperial),* G. Hammett *(PPPL)*, J. Maron *(AMNH),*
J. McWilliams *(UCLA),* S. Taylor *(Princeton)*

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The Unknown

As we know,
There are known knowns.
There are things we know we know.
We also know
There are known unknowns.
That is to say
We know there are some things
We do not know.
But there are also unknown unknowns,
The ones we don't know
We don't know.

D. H. Rumsfeld
12.02.02, DoD news briefing
as quoted by www.slate.com

Turbulence in Clusters and Galaxies

This is a cluster…

QuickTime™ and a
TIFF (Uncompressed) decompressor
are needed to see this picture.

The Coma Cluster
© NASA

Turbulence in Clusters and Galaxies

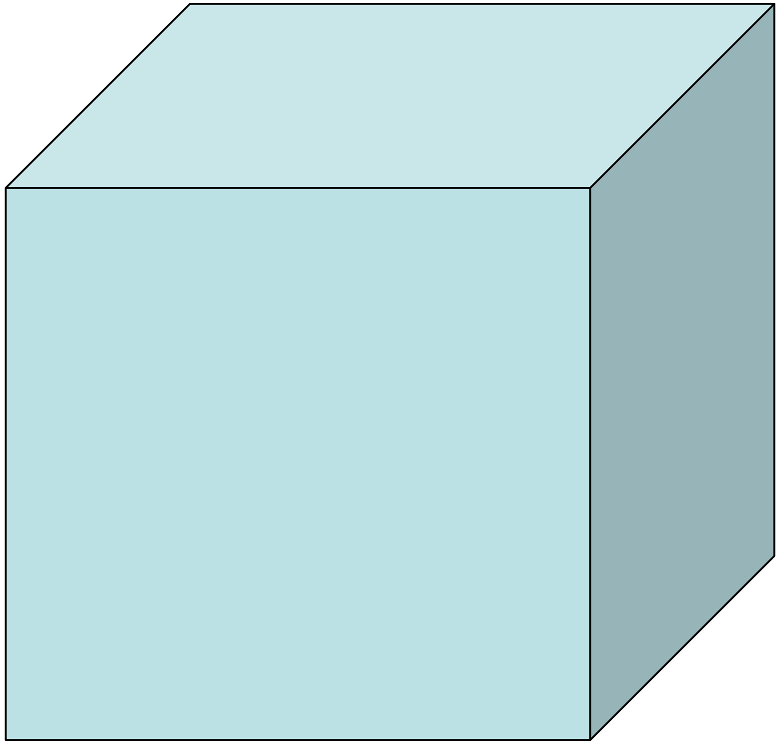
This is a galaxy…

QuickTime™ and a
TIFF (Uncompressed) decompressor
are needed to see this picture.

M51
The Whirlpool
Galaxy

MHD Turbulence: The Fundamental Problem

…and this is the Matrix

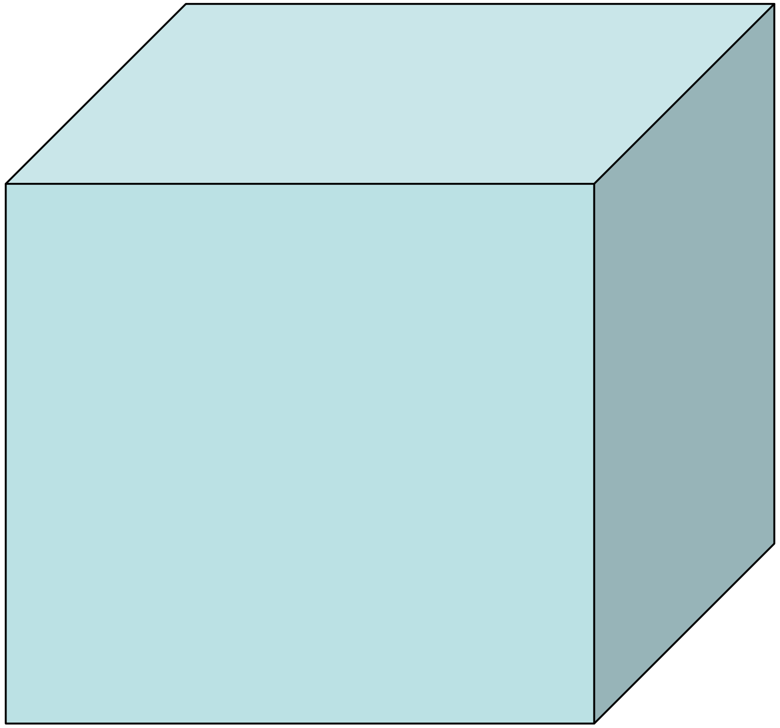


N^3
The Periodic Box

MHD Turbulence: The Fundamental Problem

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = \nu\Delta\mathbf{u} - \nabla p + \mathbf{B}\cdot\nabla\mathbf{B} + \mathbf{f}, \quad \nabla\cdot\mathbf{u} = 0$$

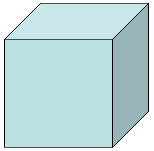
$$\partial_t \mathbf{B} + \mathbf{u}\cdot\nabla\mathbf{B} = \mathbf{B}\cdot\nabla\mathbf{u} + \eta\Delta\mathbf{B}$$



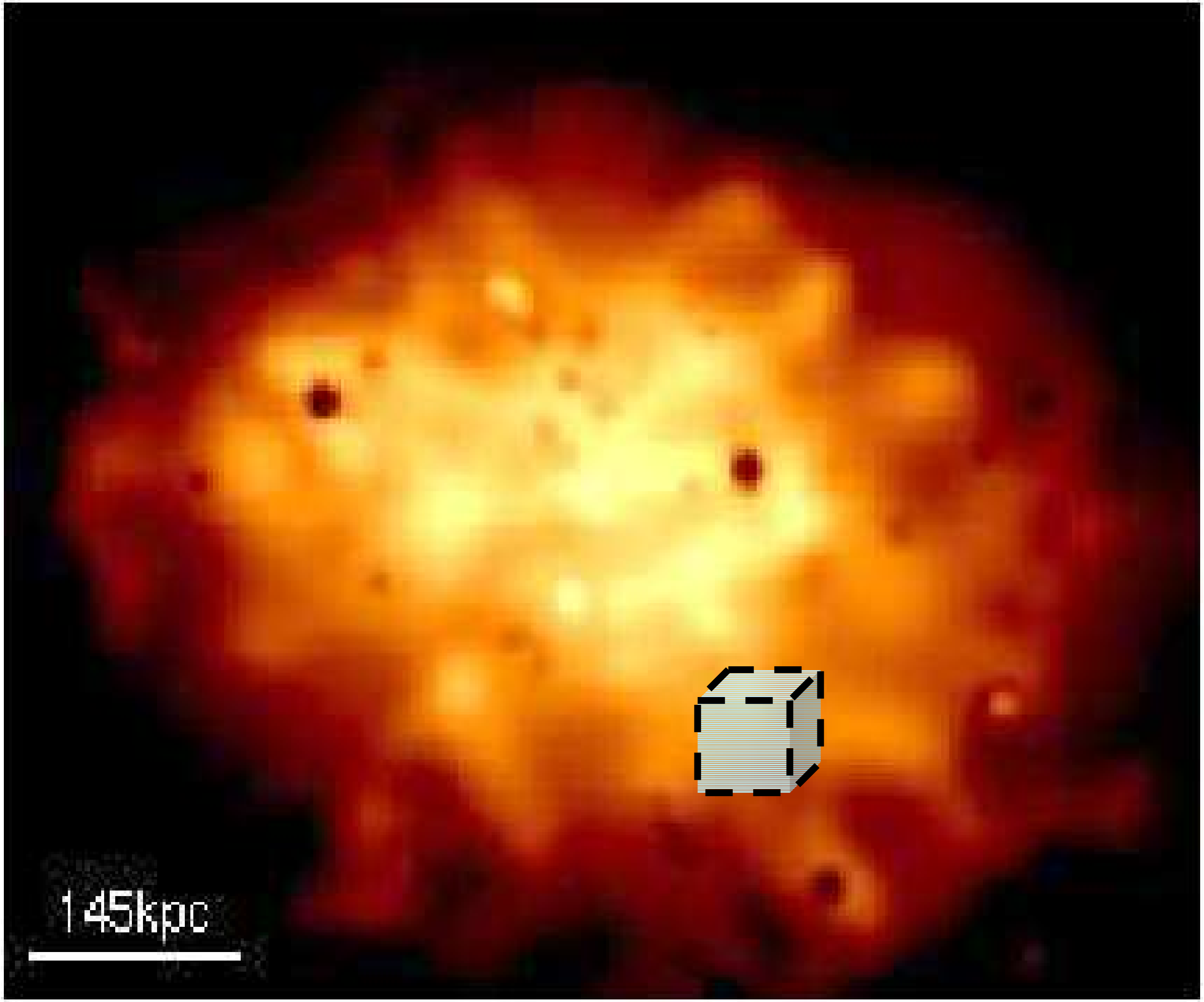
MHD Turbulence: The Fundamental Problem

$$\boxed{\begin{gathered}\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = \nu\Delta\mathbf{u} - \nabla p + \mathbf{B}\cdot\nabla\mathbf{B} + \mathbf{f}, \quad \nabla\cdot\mathbf{u} = 0 \\ \partial_t \mathbf{B} + \mathbf{u}\cdot\nabla\mathbf{B} = \mathbf{B}\cdot\nabla\mathbf{u} + \eta\Delta\mathbf{B}\end{gathered}}$$

N^3
The Periodic Box

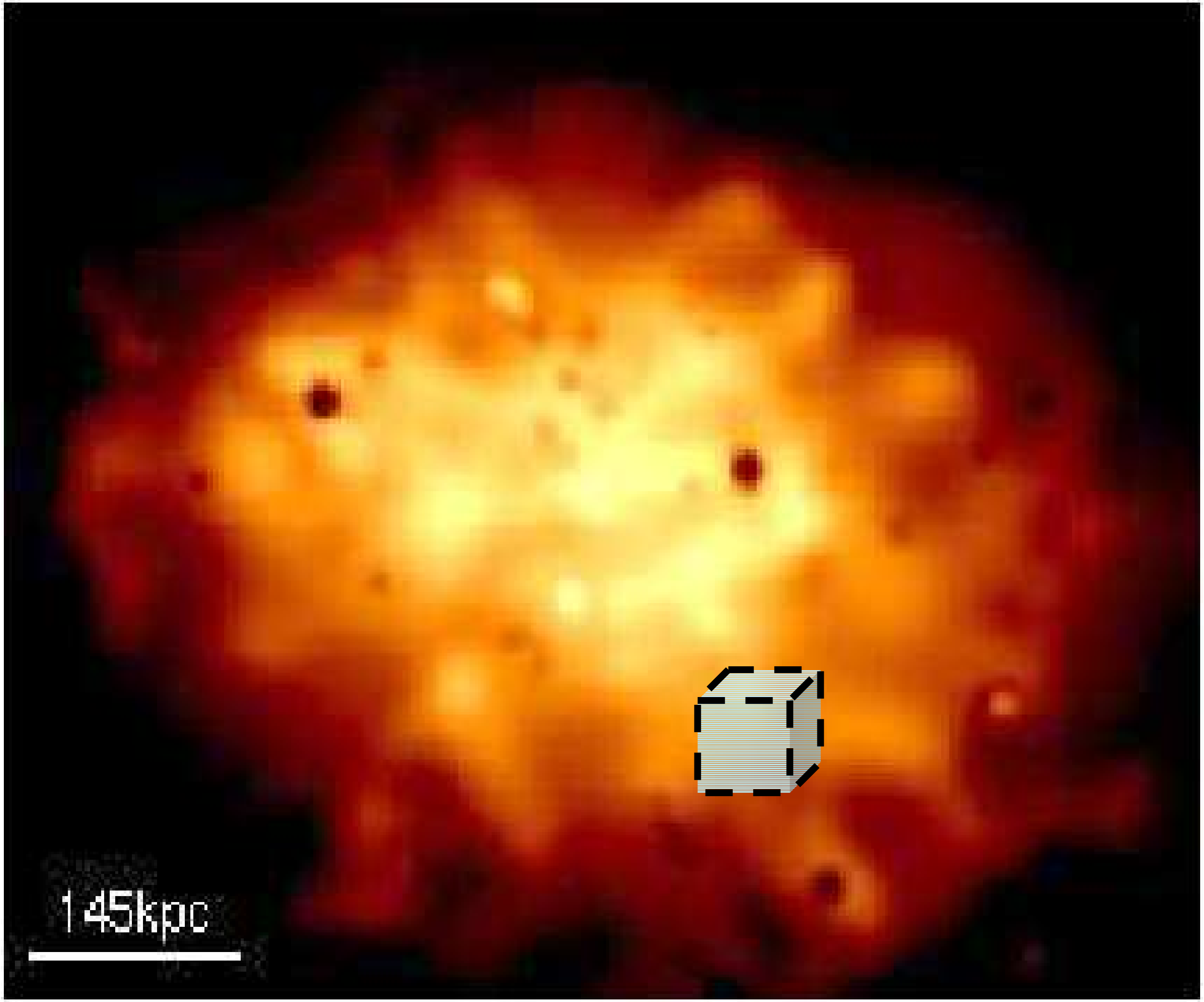


MHD Turbulence: The Fundamental Problem



The Coma Cluster [Schuecker *et al.* 2004, *A&A* **426**, 387]

Turbulence: Multiscale Disorder



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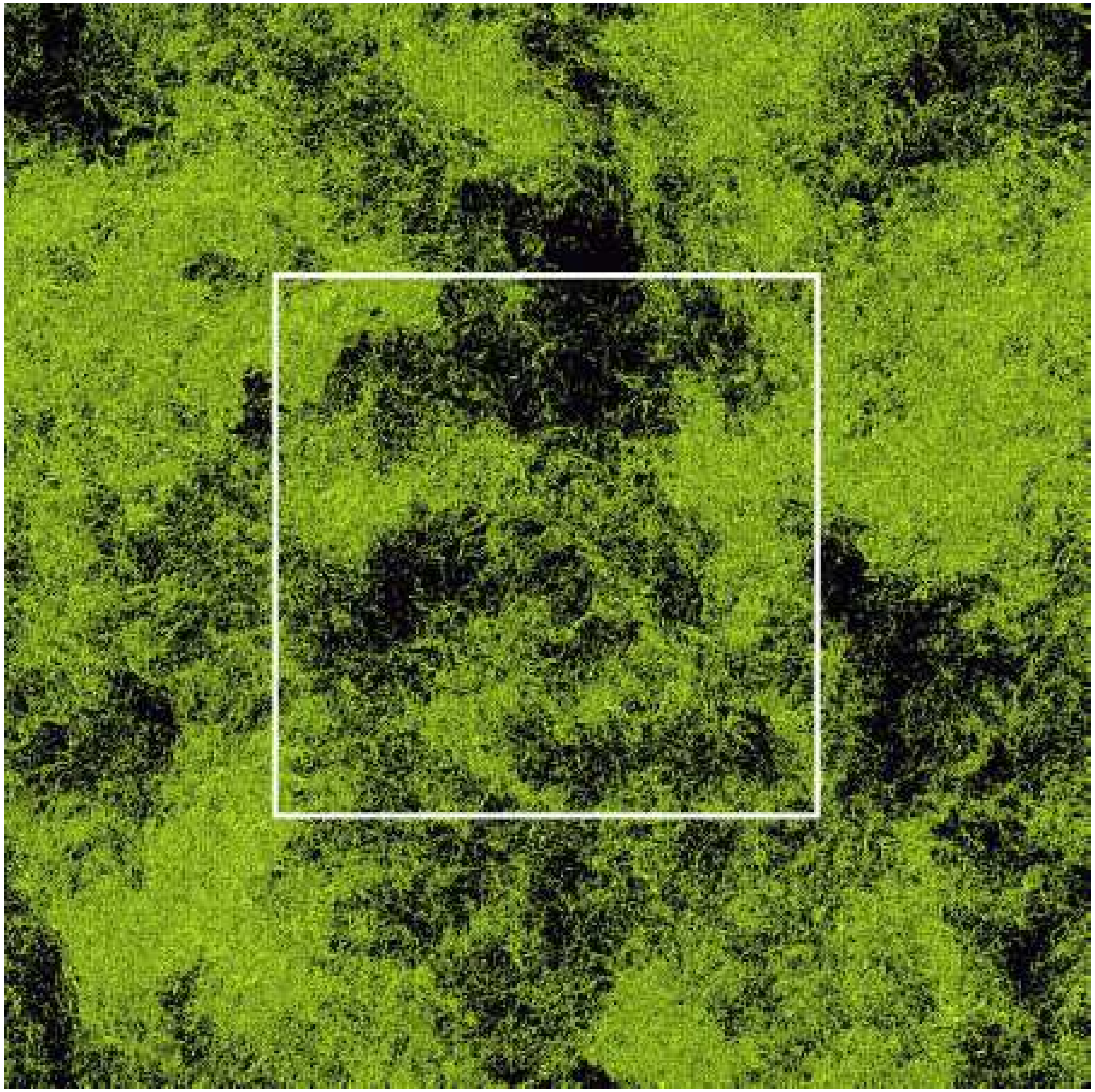
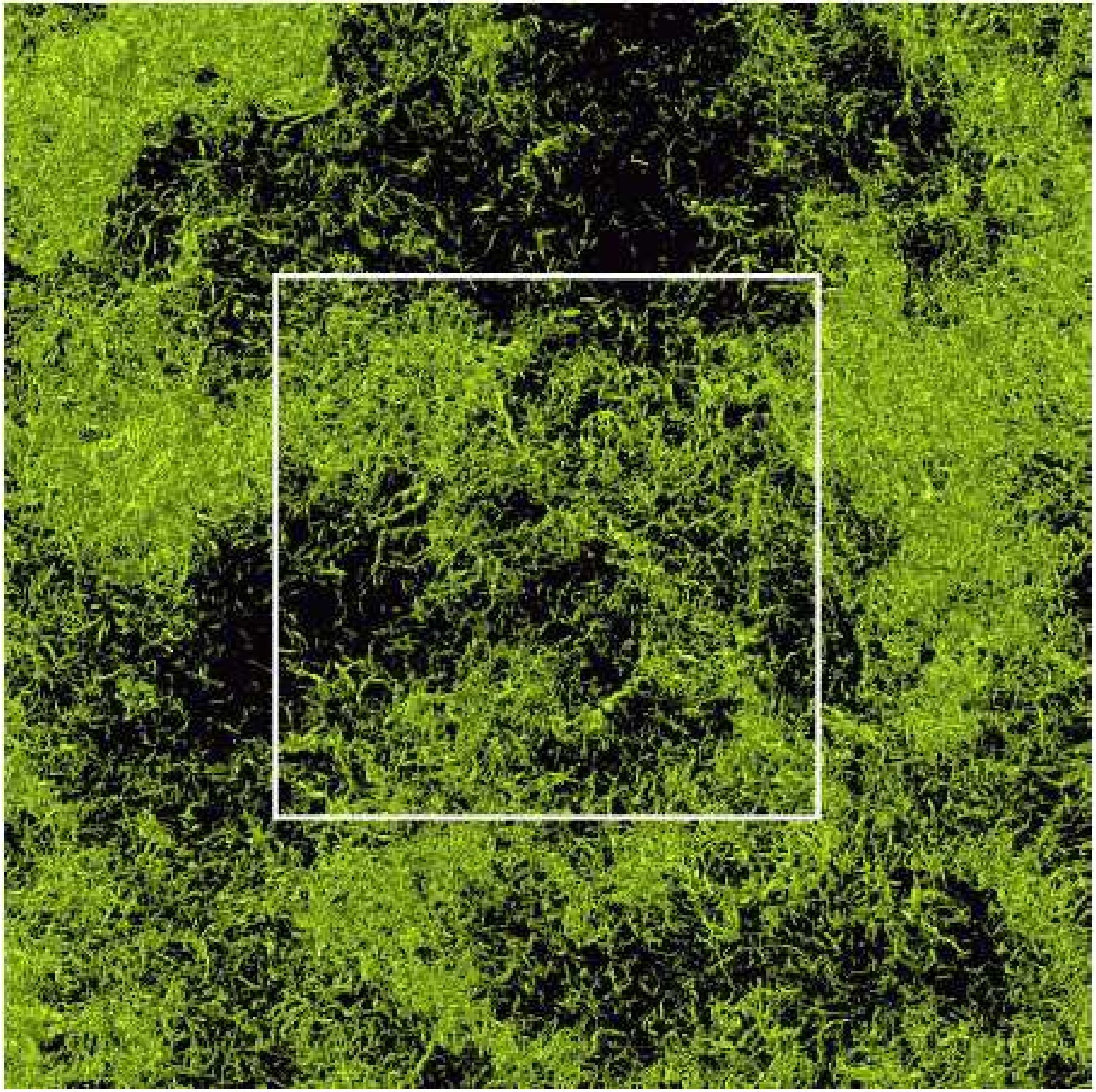


[Image: Earth Simulator, isovorticity surfaces]

Turbulence: Multiscale Disorder



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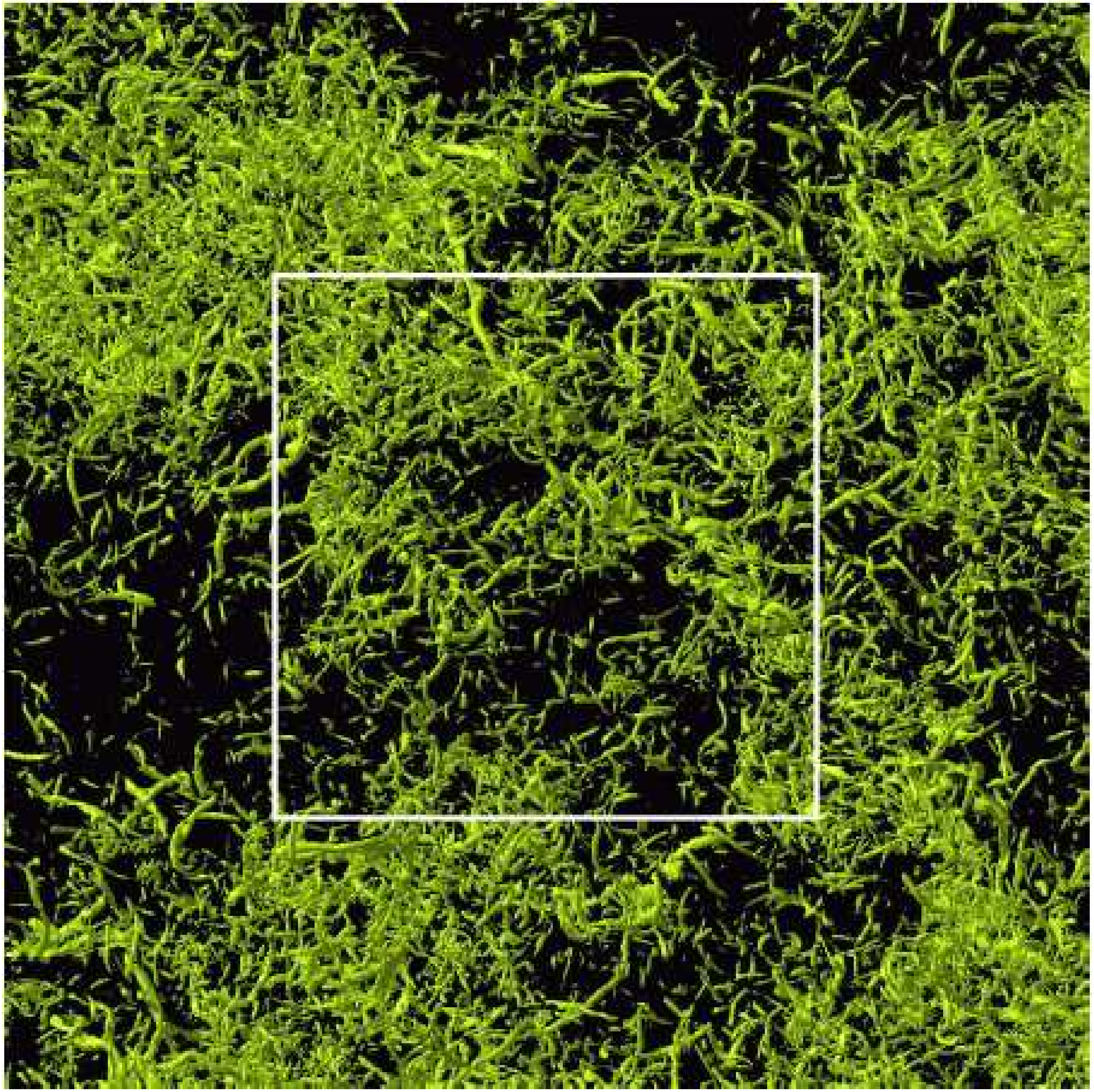
Turbulence: Multiscale Disorder



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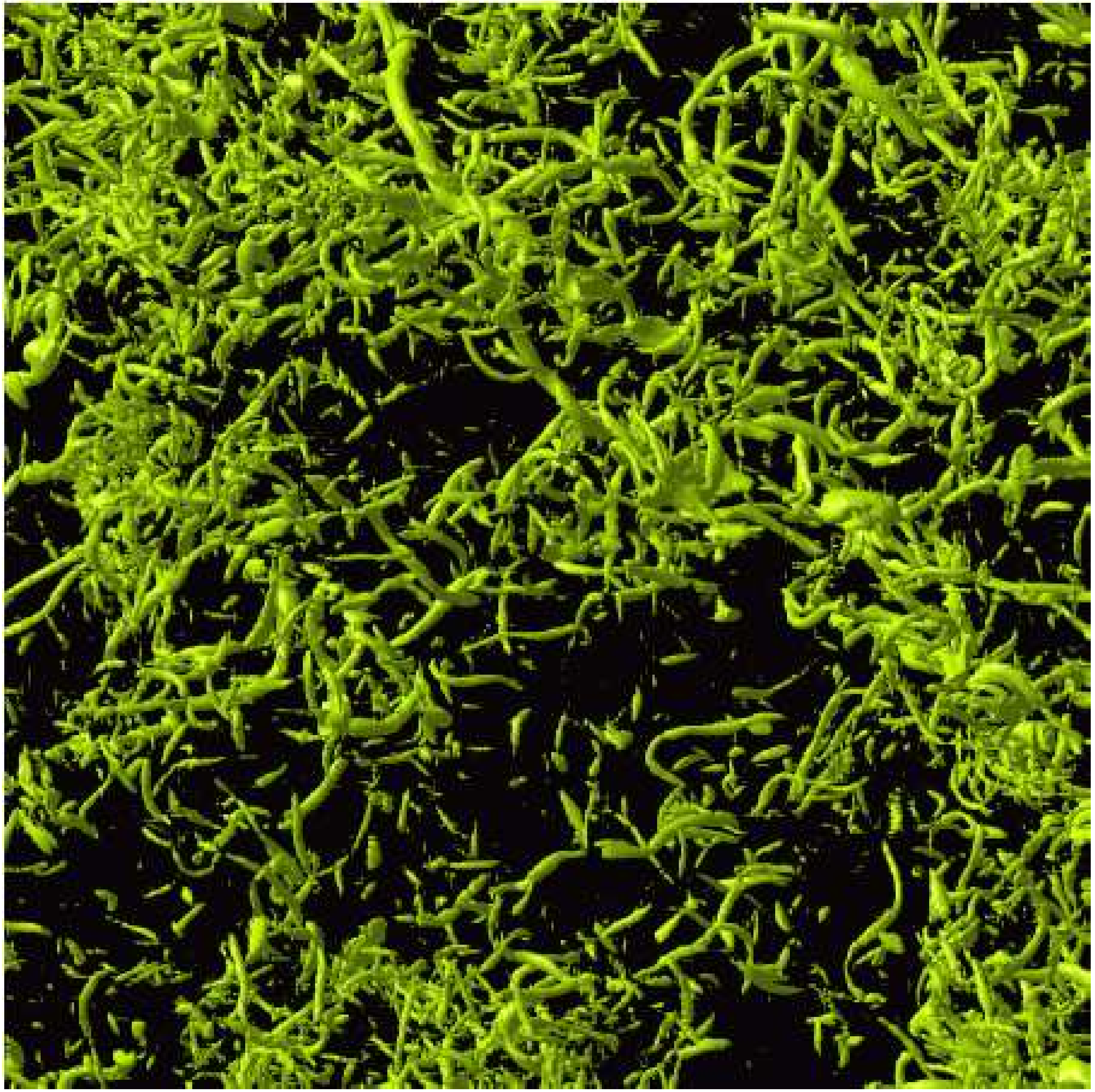
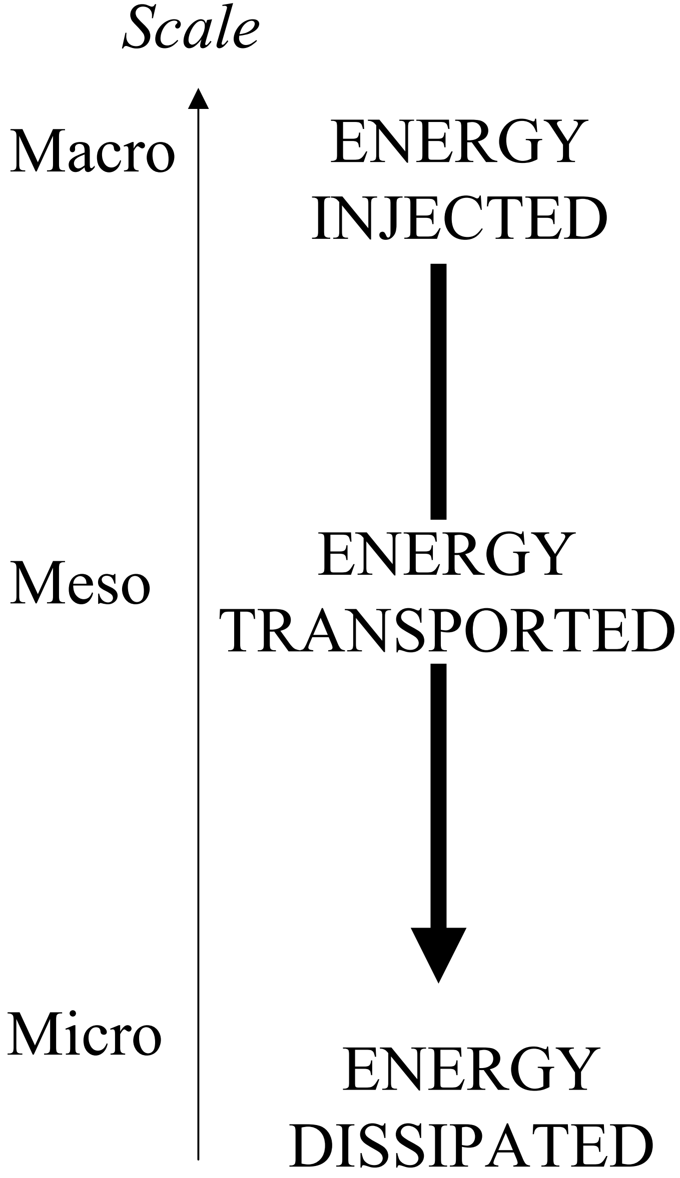
Turbulence: Multiscale Disorder



[Image: Earth Simulator, isovorticity surfaces]

Turbulence: Philosophy

Turbulence: Philosophy



Scale

Macro

ENERGY INJECTED

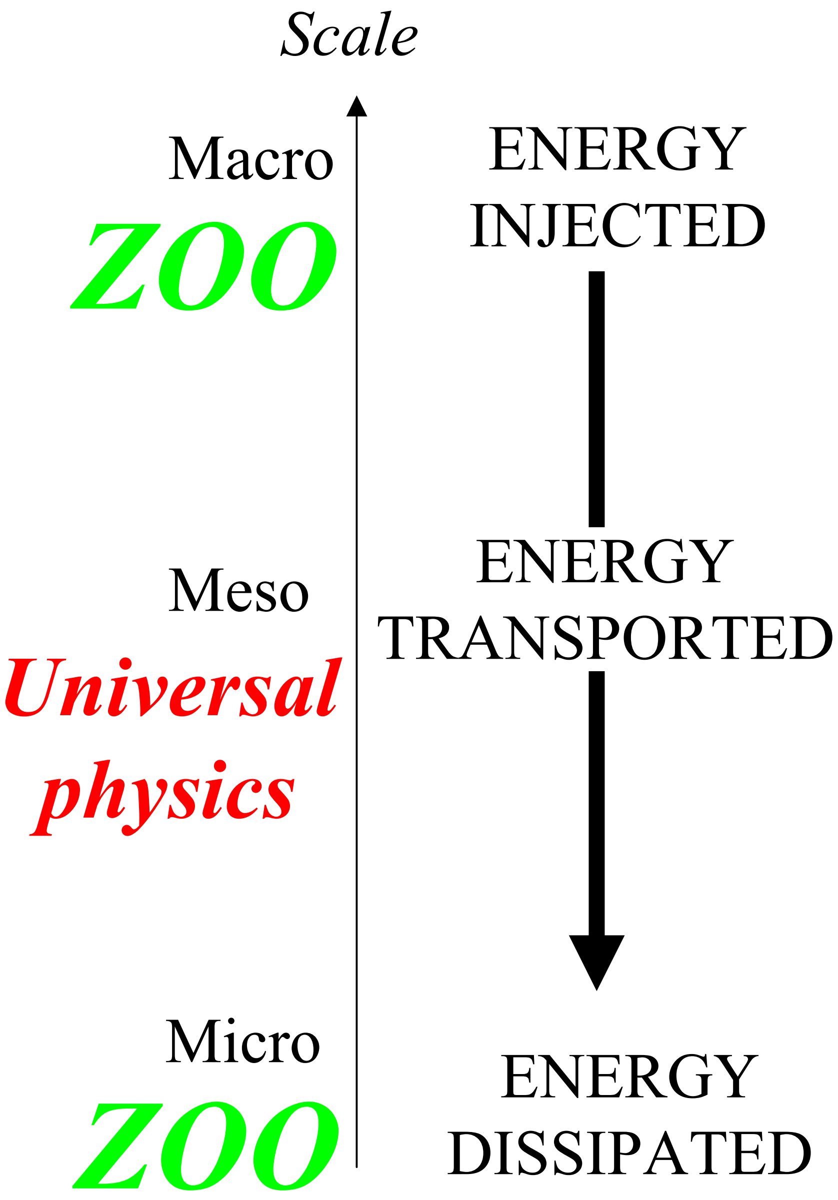
Meso

ENERGY TRANSPORTED

Micro

ENERGY DISSIPATED

Turbulence: Philosophy



Scale

Macro

ZOO

ENERGY
INJECTED

Meso

Universal physics

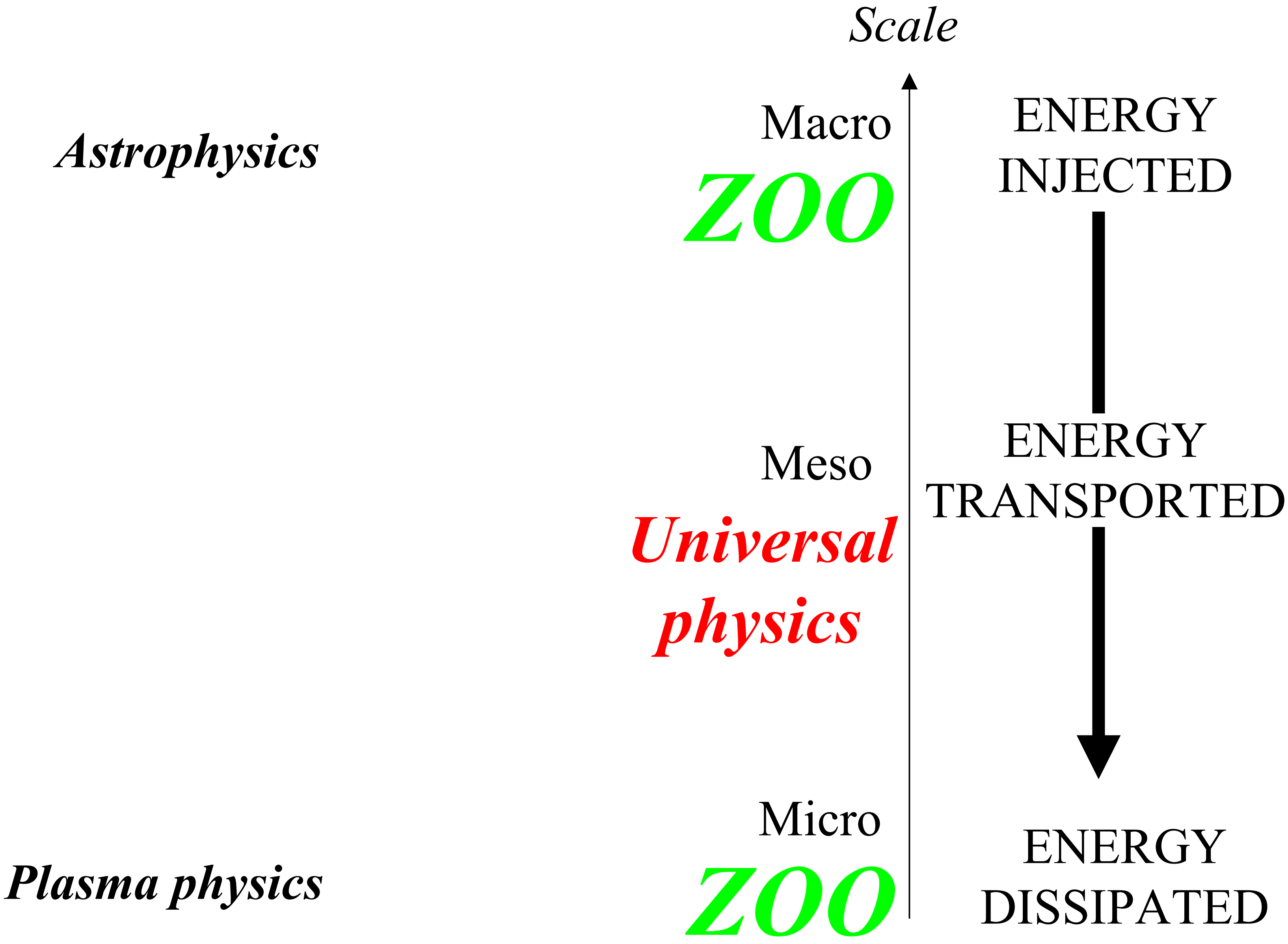
ENERGY
TRANSPORTED

Micro

ZOO

ENERGY
DISSIPATED

Turbulence: Philosophy



Scale

Astrophysics

Macro

ZOO

ENERGY INJECTED

Meso

Universal physics

ENERGY TRANSPORTED

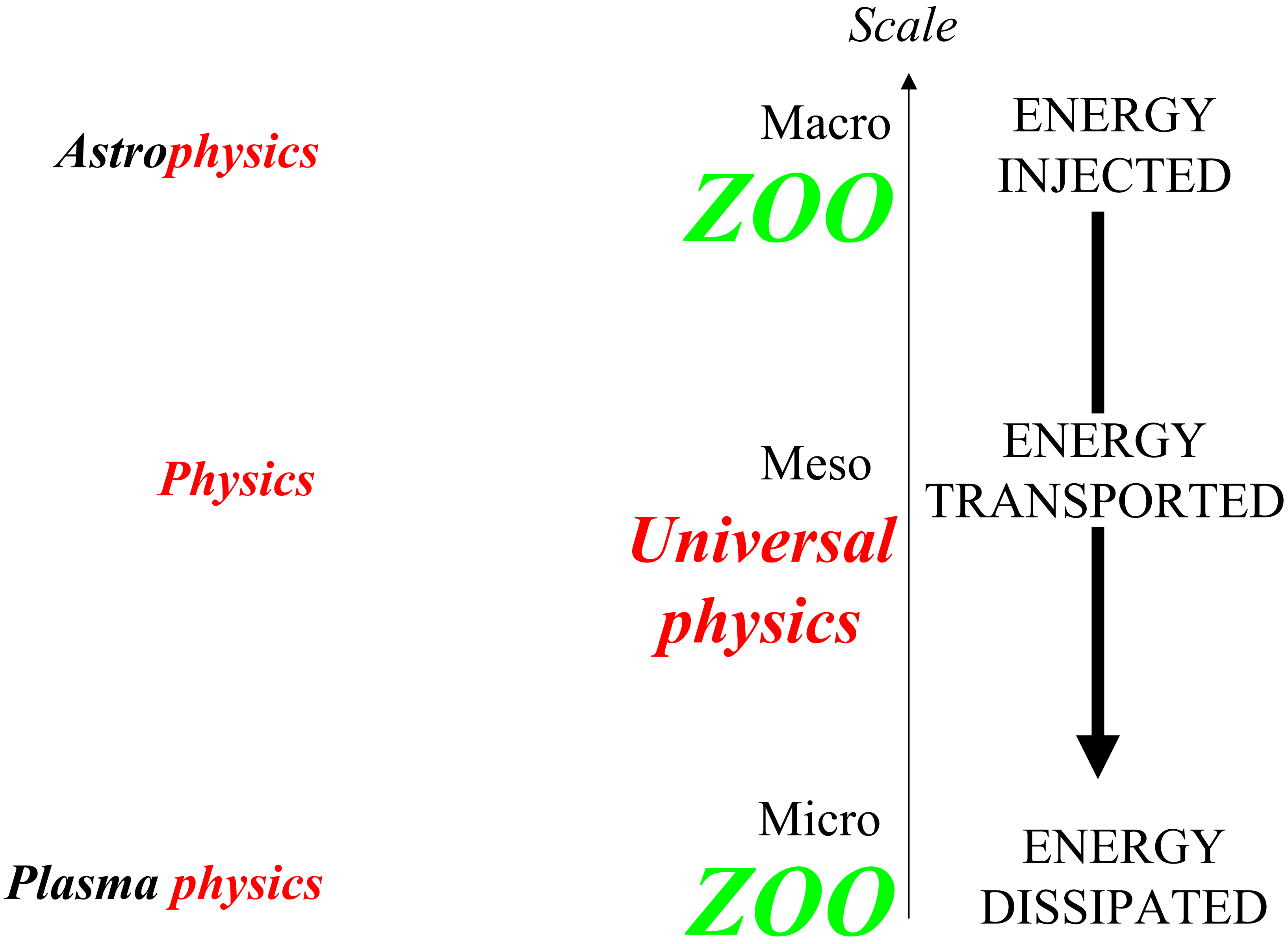
Plasma physics

Micro

ZOO

ENERGY DISSIPATED

Turbulence: Philosophy



Scale

Astrophysics

Macro

ZOO

ENERGY
INJECTED

Physics

Meso

Universal physics

ENERGY
TRANSPORTED

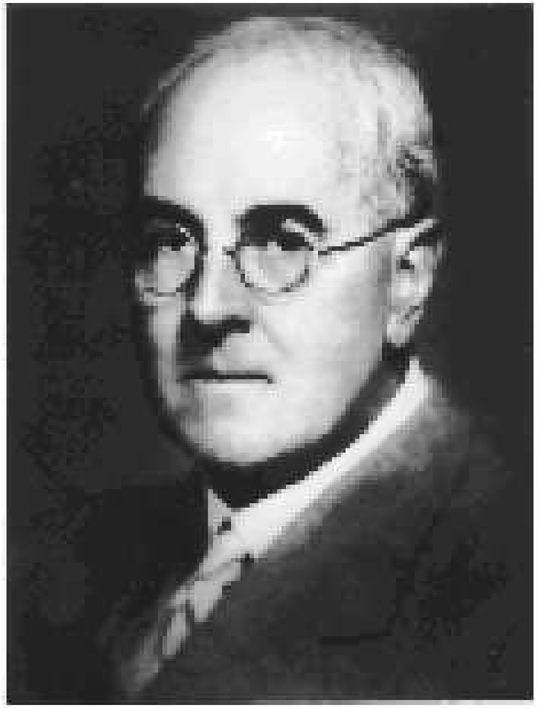
Plasma physics

Micro

ZOO

ENERGY
DISSIPATED

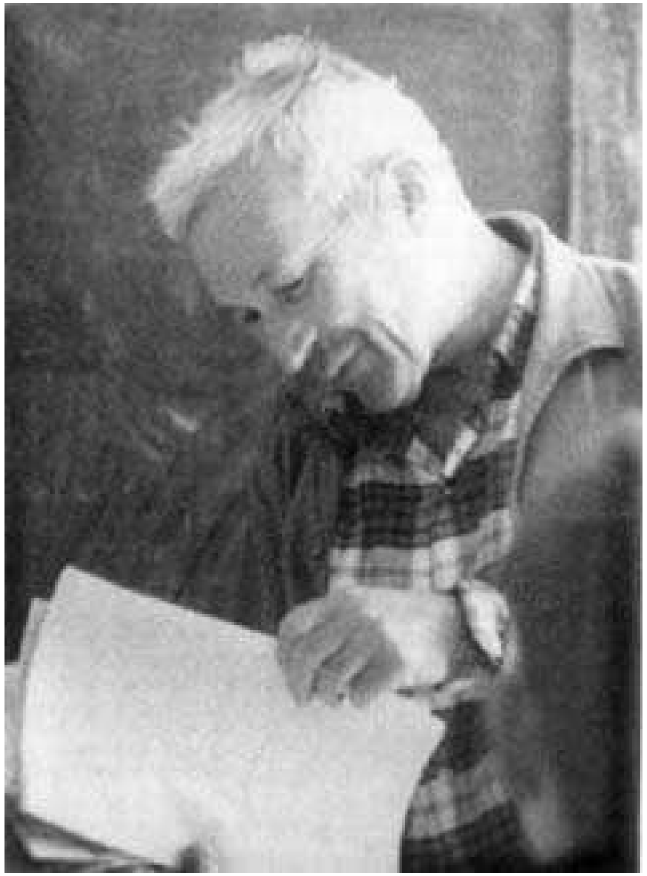
The Richardson Cascade



Big whorls have little whorls
  That feed on their velocity,
And little whorls have lesser whorls
  And so on to viscosity.

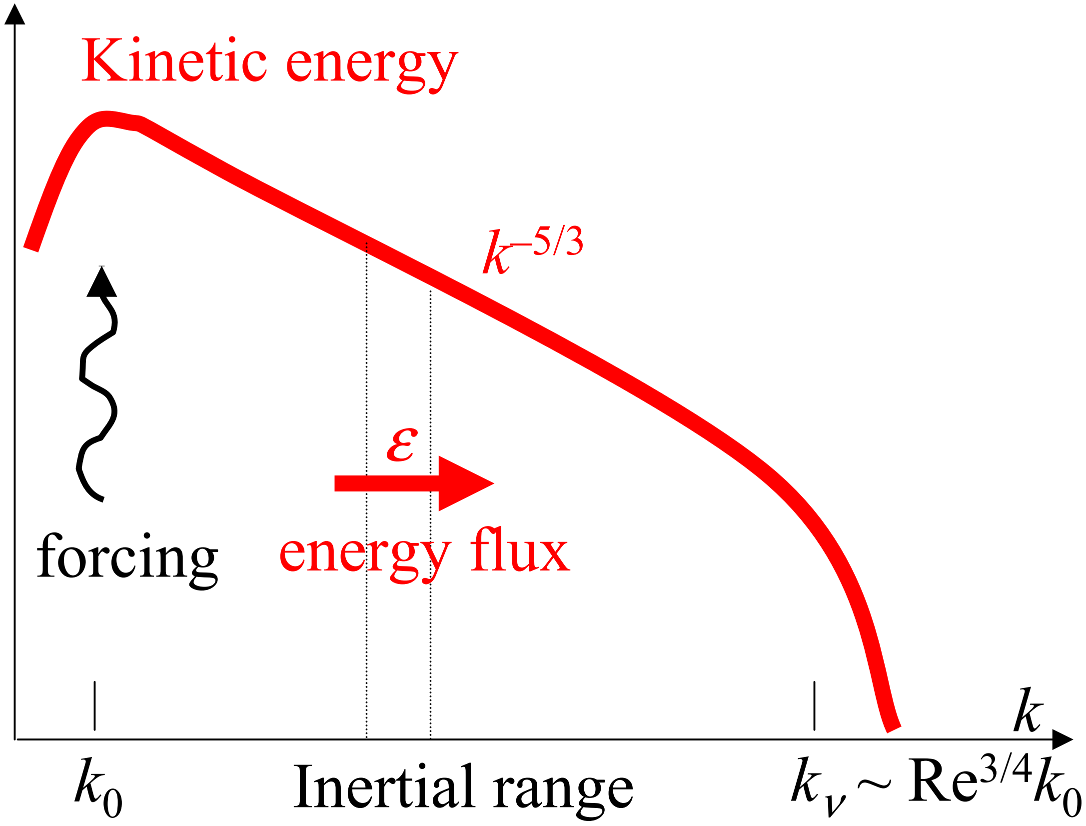
1922

Kolmogorov’s 1941 Theory



A. N. Kolmogorov
(1903-1987)

Kolmogorov Turbulence



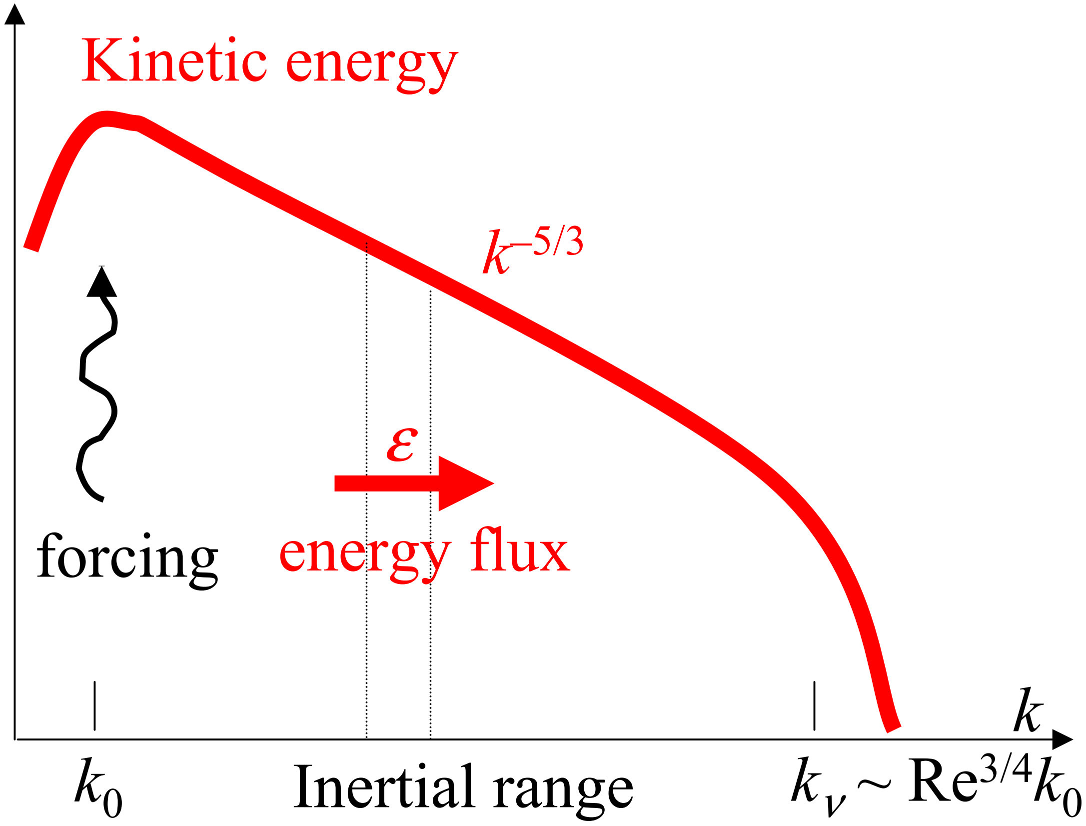
- Scale invariance
- Locality in k space

$$\varepsilon \sim u_l^2 \, \tau_l^{-1} = \text{const}$$

Energy at scale l

Cascade time (rate of transfer)

Kolmogorov Turbulence



- Scale invariance
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$$\longrightarrow \varepsilon \sim u_l^2 \, \tau_l^{-1} = \mathrm{const}$$

Energy at scale l

Cascade time (rate of transfer)

Only one time scale available at each l: the eddy-turnover time

$$\tau_l \sim \tau_{\mathrm{eddy}} \sim l/u_l \longrightarrow u_l \sim \varepsilon^{1/3} \, l^{1/3} \qquad \boxed{E(k) \sim \varepsilon^{2/3} \, k^{-5/3}}$$

K41

Kolmogorov spectrum fixed by dimensional analysis

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- **Is there universality?**
- **Are interactions local?**
- **Are macro scales important?**
- **Are micro scales important?**
- **Are magnetic and velocity fluctuations in scale-by-scale equilibrium with each other?**

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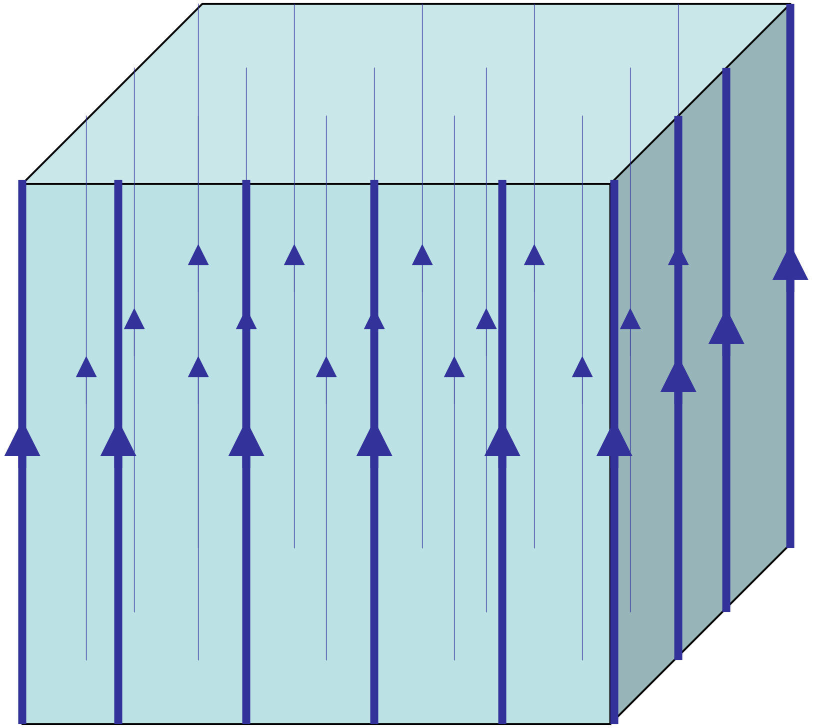
↓

- **How are energy, momentum, heat transported?**
- **How are observed magnetic fields generated and shaped?** ***(Magnetogenesis)***

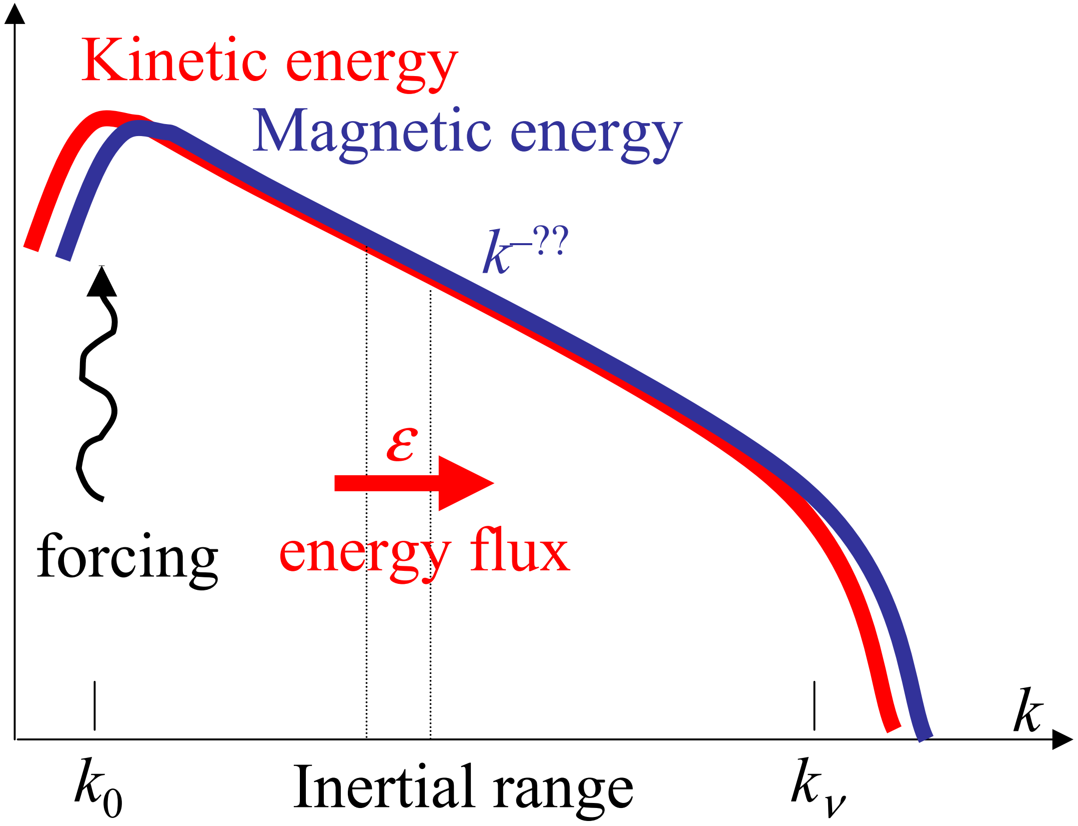
MHD Turbulence: The Fundamental Problem

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = \nu \Delta \mathbf{u} - \nabla p + \mathbf{B} \cdot \nabla \mathbf{B} + \mathbf{f}, \quad \nabla \cdot \mathbf{u} = 0$$
$$\partial_t \mathbf{B} + \mathbf{u} \cdot \nabla \mathbf{B} = \mathbf{B} \cdot \nabla \mathbf{u} + \eta \Delta \mathbf{B}$$

Strong uniform
mean field $\boldsymbol{B}_0$
imposed
$B_0 > u_{\mathrm{rms}}$



MHD Turbulence à la Kolmogorov



- **Strong mean field B_0**
- **Alfvénic state: $u_l \sim B_l$**
- Scale invariance
- Locality in k space

$$\varepsilon \sim u_l^2\, \tau_l^{-1} = \text{const}$$

Energy at scale l

Cascade time (rate of transfer)

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = \nu\Delta\mathbf{u} - \nabla p + \mathbf{B}\cdot\nabla\mathbf{B} + \mathbf{f}, \quad \nabla\cdot\mathbf{u} = 0$$

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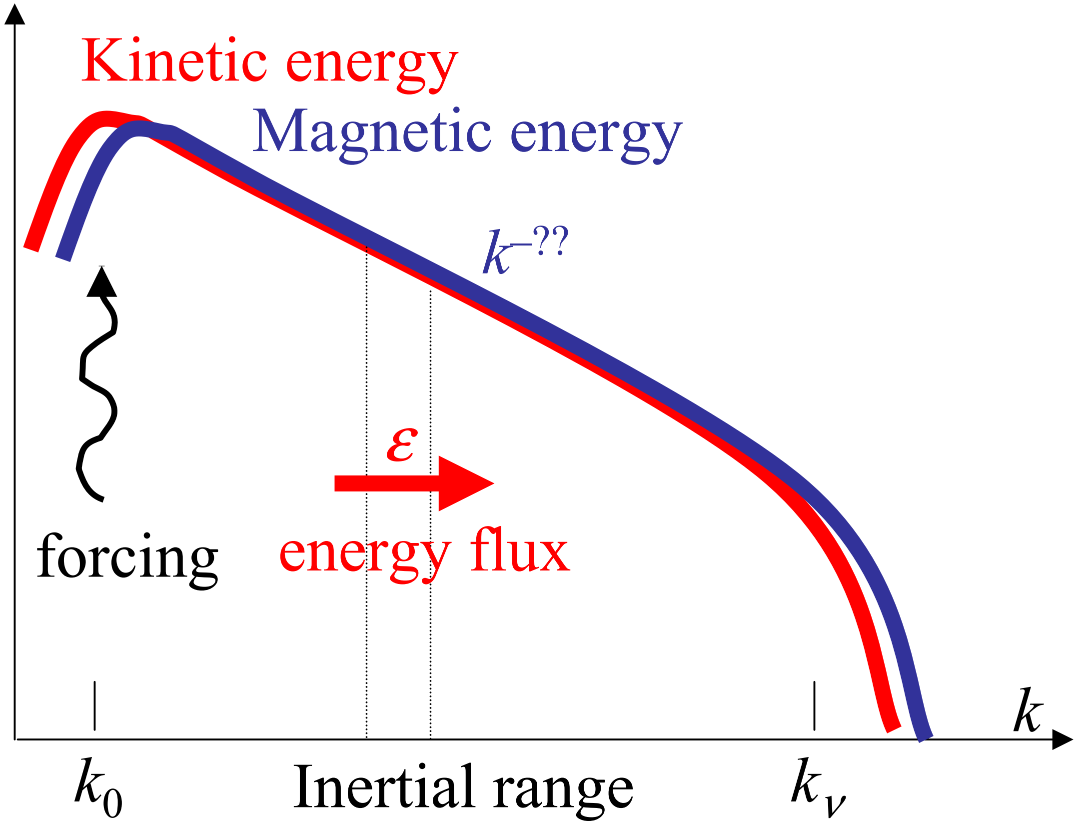

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Energy at scale l

Cascade time (rate of transfer)

Two time scales available:

turnover time: $\tau_{\text{eddy}} \sim l_\perp / u_l$

Alfvén time: $\tau_{\text{A}} \sim l_{||} / v_{\text{A}}$

$v_{\text{A}} = B_0 / (4\pi\rho)^{1/2}$

$\boxed{\tau_l \sim ?}$

Cannot fix scalings solely by dimensional analysis!

Interaction of Alfvén Wave Packets — I

$$\text{Elsasser fields}\quad \mathbf{z}^{\pm} = \mathbf{u} \pm \delta\mathbf{B}$$

$$\frac{\partial \mathbf{z}^{\pm}}{\partial t} \mp v_{\mathrm{A}} \frac{\partial \mathbf{z}^{\pm}}{\partial z} + \mathbf{z}^{\mp} \cdot \nabla \mathbf{z}^{\pm} = -\nabla p + \frac{\nu + \eta}{2} \Delta \mathbf{z}^{\pm} + \frac{\nu - \eta}{2} \Delta \mathbf{z}^{\mp}$$

[Iroshnikov 1964, *Sov. Astron.* **7**, 566; Kraichnan 1965, *Phys. Fluids* **8**, 1385]

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Only counterpropagating waves interact

[Iroshnikov 1964, *Sov. Astron.* **7**, 566; Kraichnan 1965, *Phys. Fluids* **8**, 1385]

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Assume weak interactions: $\tau_{\mathrm{eddy}} >> \tau_{\mathrm{A}}$

- Wave packet passes through another: $\delta t \sim \dfrac{l_{\|}}{v_{\mathrm{A}}} \sim \tau_{\mathrm{A}}$

[Iroshnikov 1964, *Sov. Astron.* **7**, 566; Kraichnan 1965, *Phys. Fluids* **8**, 1385]

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- Wave packet passes through another: $\delta t\sim\frac{l_{\|}}{v_{\mathrm{A}}}\sim\tau_{\mathrm{A}}$

- Its amplitude gets a kick: $\delta u_l\sim\frac{u_l^2}{l}\delta t\sim u_l\frac{u_l}{l}\frac{l_{\|}}{v_{\mathrm{A}}}\sim u_l\frac{\tau_{\mathrm{A}}}{\tau_{\mathrm{eddy}}}$

[Iroshnikov 1964, *Sov. Astron.* **7**, 566; Kraichnan 1965, *Phys. Fluids* **8**, 1385]

Interaction of Alfvén Wave Packets — I

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- Sum of kicks over time *t*: $\sum^{t} \delta u_l \sim u_l \frac{\tau_{\mathrm{A}}}{\tau_{\mathrm{eddy}}} \sqrt{\frac{t}{\tau_{\mathrm{A}}}}$

[Iroshnikov 1964, *Sov. Astron.* **7**, 566; Kraichnan 1965, *Phys. Fluids* **8**, 1385]

Interaction of Alfvén Wave Packets — I

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Assume weak interactions: $\tau_{\rm eddy} >> \tau_{\rm A}$

- Wave packet passes through another: $\delta t \sim \frac{l_{\|}}{v_{\rm A}} \sim \tau_{\rm A}$
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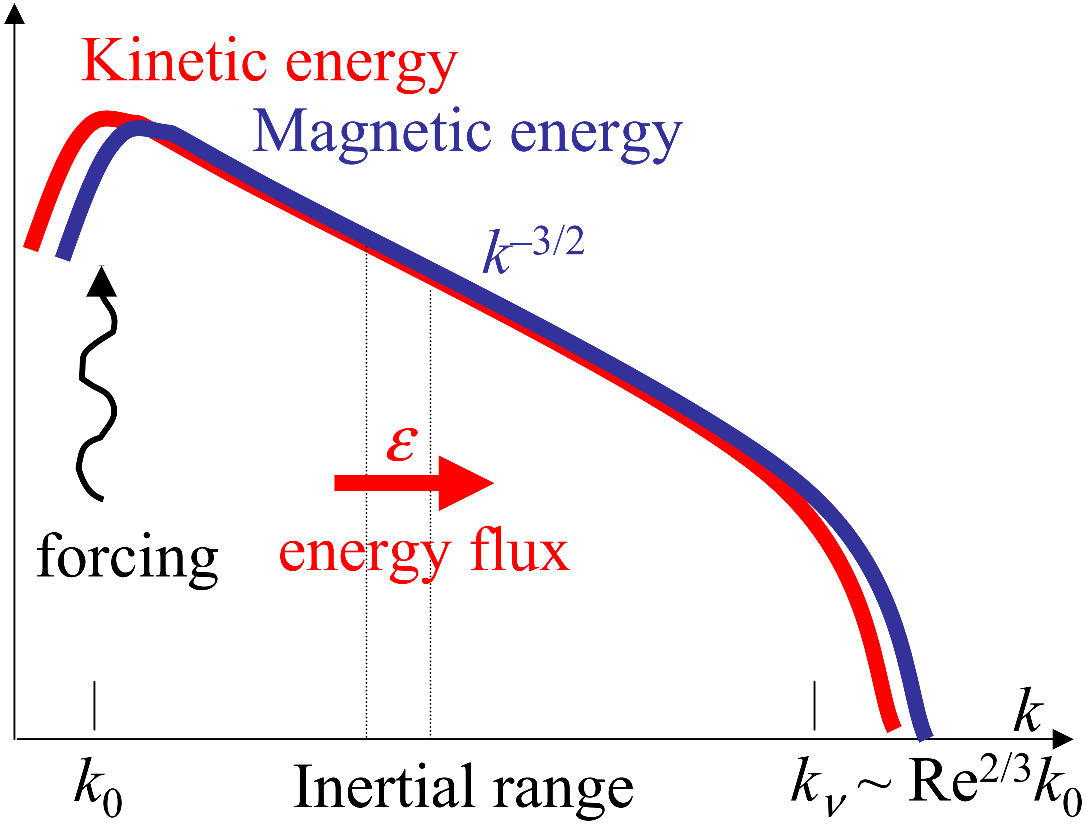
- Sum of kicks over time *t*: $\sum^{t} \delta u_l \sim u_l \frac{\tau_{\rm A}}{\tau_{\rm eddy}}\sqrt{\frac{t}{\tau_{\rm A}}}$
- Cascade time: $t \sim \tau_l \;\Leftrightarrow\; \sum^{t} \delta u_l \sim u_l \;\Rightarrow\; \boxed{\tau_l \sim \frac{\tau_{\rm eddy}^2}{\tau_{\rm A}}}$

[Iroshnikov 1964, *Sov. Astron.* **7**, 566; Kraichnan 1965, *Phys. Fluids* **8**, 1385]

Iroshnikov-Kraichnan Turbulence



- **Strong mean field B_0**
- **Alfvénic state: $u_l \sim B_l$**
- Scale invariance
- Locality in k space

$$\varepsilon \sim u_l^2\, \tau_l^{-1} = \mathrm{const}$$

Energy at scale l

Cascade time (rate of transfer)

Additional physical assumptions:

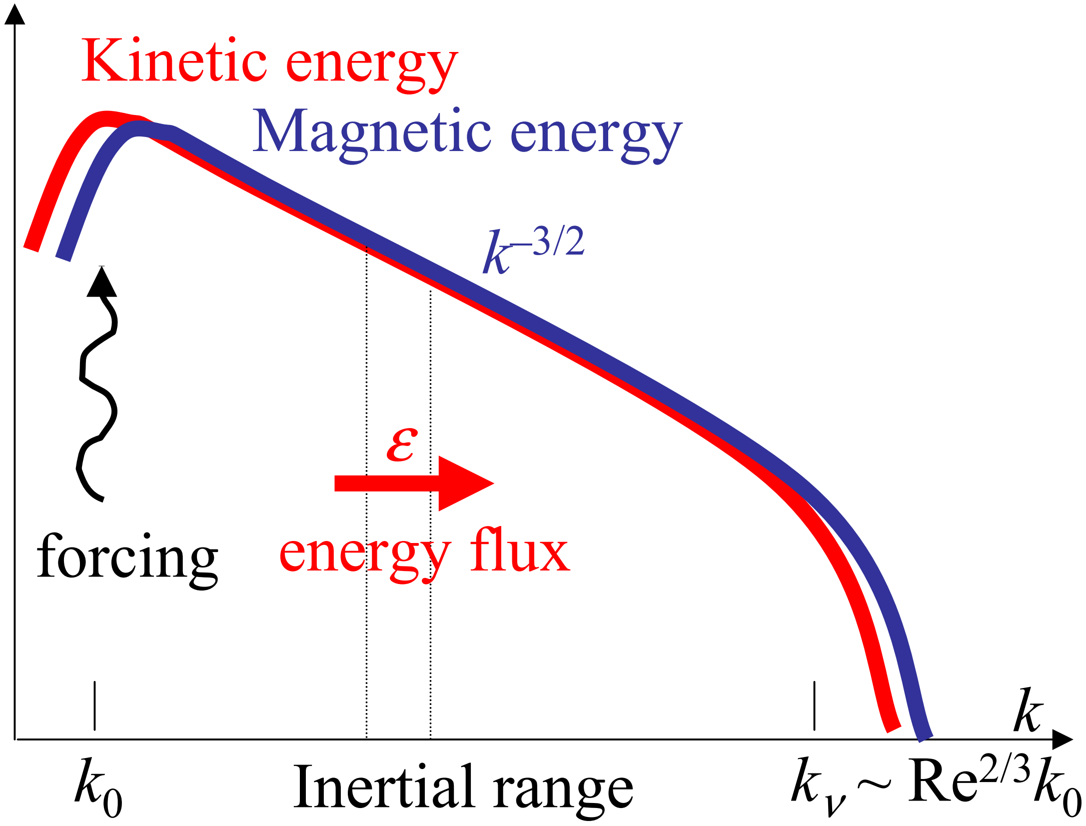
- **weak interactions: $\tau_{\mathrm{eddy}} >> \tau_A$**

$$\tau_l \sim \frac{\tau_{\mathrm{eddy}}^2}{\tau_A} \sim \frac{l_\perp^2}{u_l^2}\frac{v_A}{l_\parallel}$$

$$\boxed{u_l \sim (\varepsilon v_A)^{1/4} l_\perp^{1/2} l_\parallel^{-1/4}}$$

[Iroshnikov 1964, *Sov. Astron.* **7**, 566; Kraichnan 1965, *Phys. Fluids* **8**, 1385]

Iroshnikov-Kraichnan Turbulence



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Energy at scale l

Cascade time (rate of transfer)

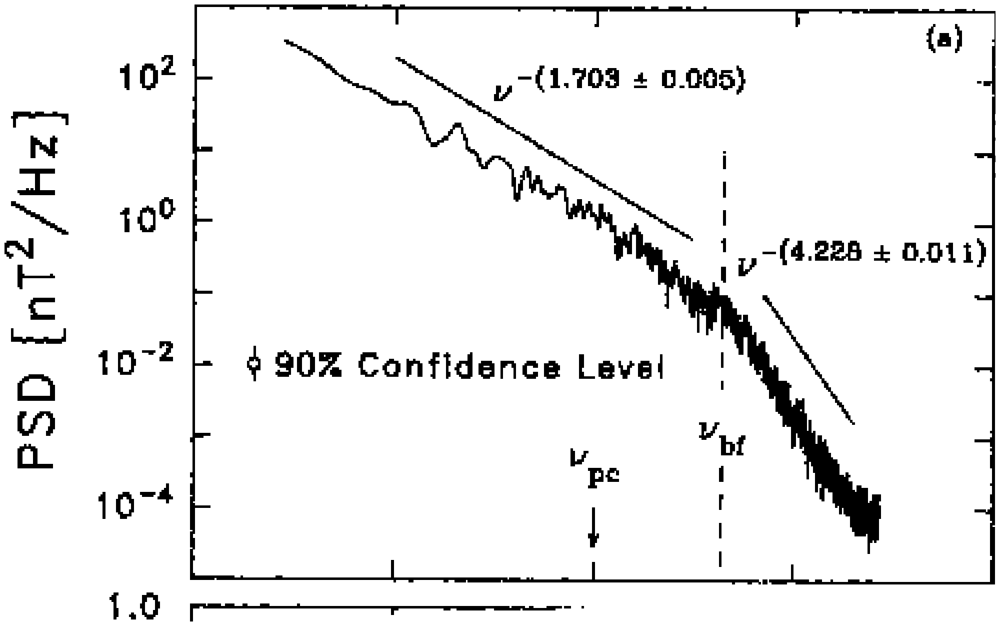
Additional physical assumptions:

- **weak interactions:** $\tau_{\text{eddy}} >> \tau_A$
- **isotropy:** $l_\| \sim l_\perp$

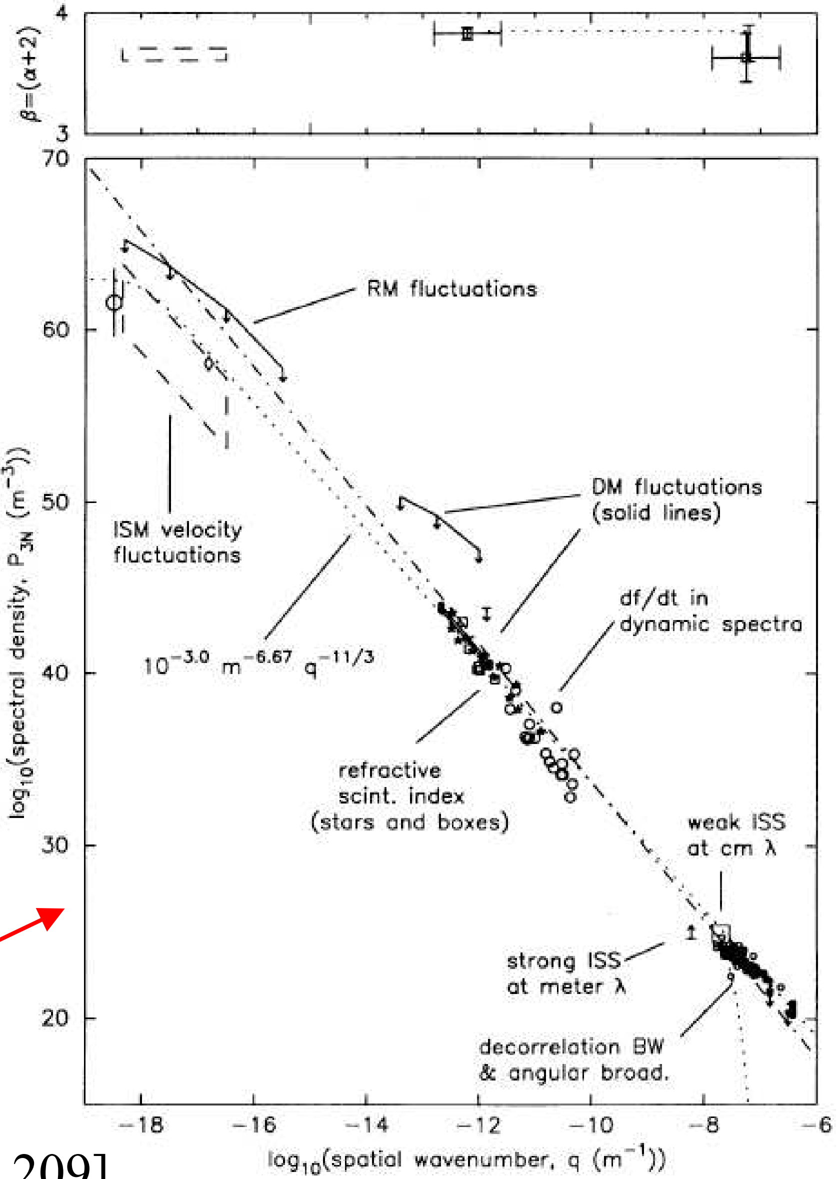
$$\longrightarrow \boxed{E(k) \sim (\varepsilon v_A)^{1/2} \, k^{-3/2}}$$ **IK65**

[Iroshnikov 1964, *Sov. Astron.* **7**, 566; Kraichnan 1965, *Phys. Fluids* **8**, 1385]

Observations: Spectrum is not $k^{-3/2}$?

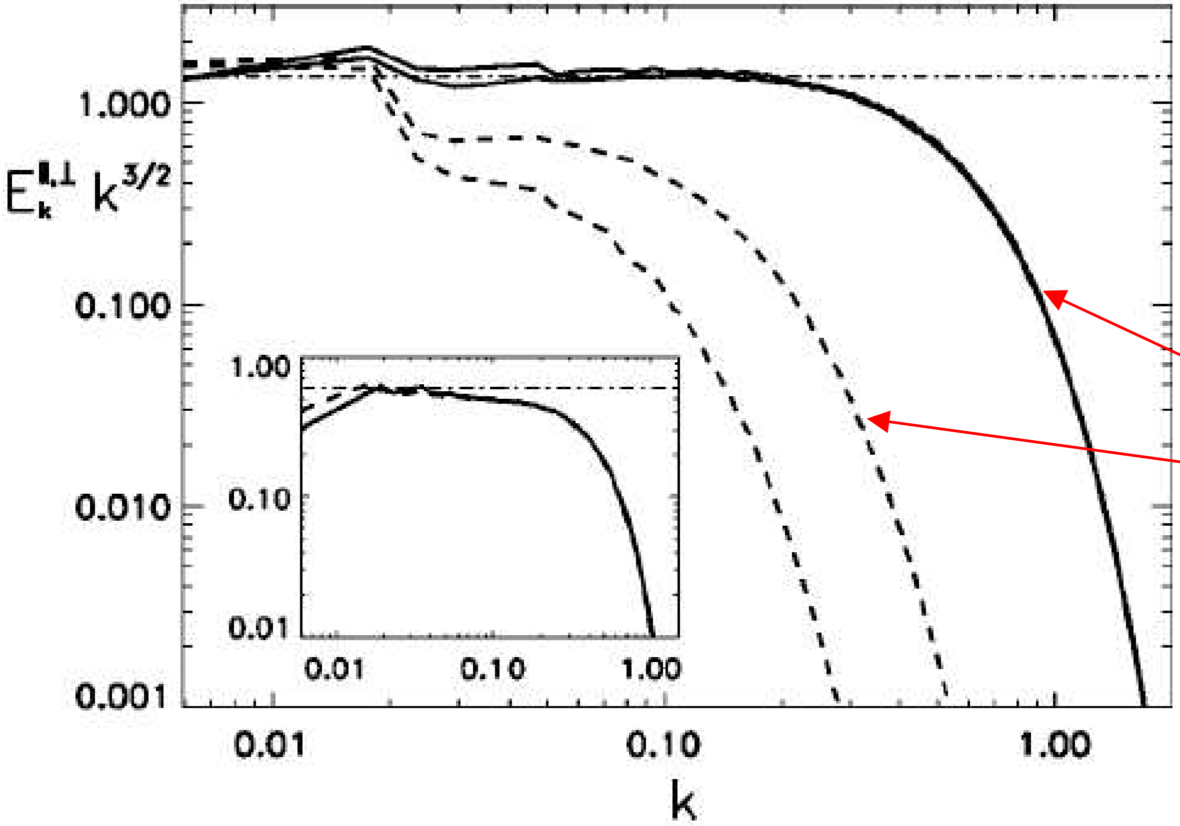


Solar wind
[Leamon *et al.* 1998]



Electron density in the ISM
[Armstrong *et al*. 1995, *ApJ* **443**, 209]

DNS: MHD Turbulence is Anisotropic!

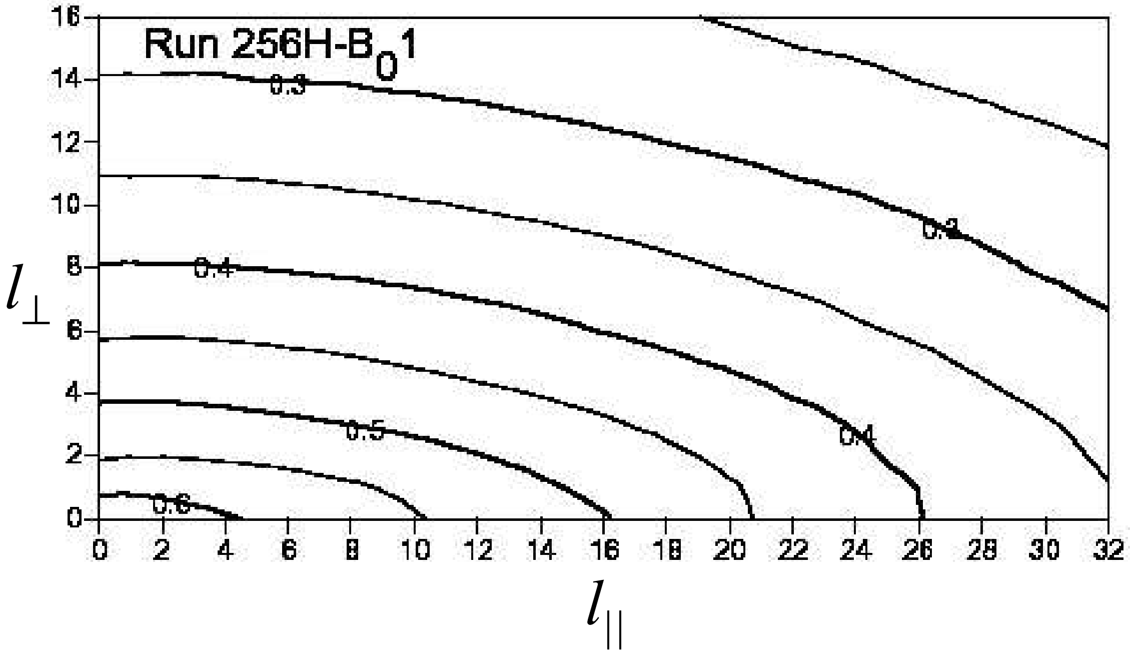


Müller *et al.* 2003, *PRE* **67**, 066302: perpendicular and parallel spectra

- **weak interactions: $\tau_{\text{eddy}} >> \tau_A$**
- **isotropy: $l_\parallel \sim l_\perp$**

$$\longrightarrow \boxed{E(k) \sim (\varepsilon v_A)^{1/2}\, k^{-3/2}}$$ **IK65**

DNS: MHD Turbulence is Anisotropic!



Cho *et al*. 2002,
ApJ **564**, 291:
contours of velocity
correlation functions

- **weak interactions: $\tau_{\mathrm{eddy}} >> \tau_{\mathrm{A}}$**
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$\longrightarrow$ $\boxed{E(k) \sim (\varepsilon v_A)^{1/2}\, k^{-3/2}}$ **IK65**

Interaction of Alfvén Wave Packets — II

Elsasser fields $\mathbf{z}^{\pm} = \mathbf{u} \pm \delta\mathbf{B}$

$$\frac{\partial \mathbf{z}^{\pm}}{\partial t} \mp v_{\mathrm{A}} \frac{\partial \mathbf{z}^{\pm}}{\partial z} + \mathbf{z}^{\mp} \cdot \nabla \mathbf{z}^{\pm} = -\nabla p + \frac{\nu + \eta}{2} \Delta \mathbf{z}^{\pm} + \frac{\nu - \eta}{2} \Delta \mathbf{z}^{\mp}$$

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Only counterpropagating waves interact:

$$\omega(\mathbf{k}) = \pm k_{\|} v_{\mathrm{A}}$$

$$\begin{array}{ccccc} \omega(\mathbf{k}_1) + \omega(\mathbf{k}_2) = \omega(\mathbf{k}_3) & \longrightarrow & k_{\|1} - k_{\|2} = k_{\|3} & & k_{\|2} = 0 \\ & & & \longrightarrow & \\ \mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 & \longrightarrow & k_{\|1} + k_{\|2} = k_{\|3} & & k_{\|1} = k_{\|3} \end{array}$$

Interaction of Alfvén Wave Packets — II

$$\text{Elsasser fields}\quad \mathbf{z}^{\pm} = \mathbf{u} \pm \delta\mathbf{B}$$

$$\frac{\partial \mathbf{z}^{\pm}}{\partial t} \mp v_{\mathrm{A}} \frac{\partial \mathbf{z}^{\pm}}{\partial z} + \boxed{\mathbf{z}^{\mp} \cdot \nabla \mathbf{z}^{\pm}} = -\nabla p + \frac{\nu + \eta}{2} \Delta \mathbf{z}^{\pm} + \frac{\nu - \eta}{2} \Delta \mathbf{z}^{\mp}$$

Only counterpropagating waves interact:

$$\omega(\mathbf{k}) = \pm k_{\|} v_{\mathrm{A}}$$

$$\begin{array}{lclcl} \omega(\mathbf{k}_1) + \omega(\mathbf{k}_2) = \omega(\mathbf{k}_3) & \longrightarrow & k_{\|1} - k_{\|2} = k_{\|3} & & k_{\|2} = 0 \\ & & & \longrightarrow & \\ \mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 & \longrightarrow & k_{\|1} + k_{\|2} = k_{\|3} & & k_{\|1} = k_{\|3} \end{array}$$

- **No cascade in $k_{\|}$**
- **Alfvén waves interact via $k_{\|} = 0$ modes**

Issues With $k_{||} = 0$ ("Mean Modes")

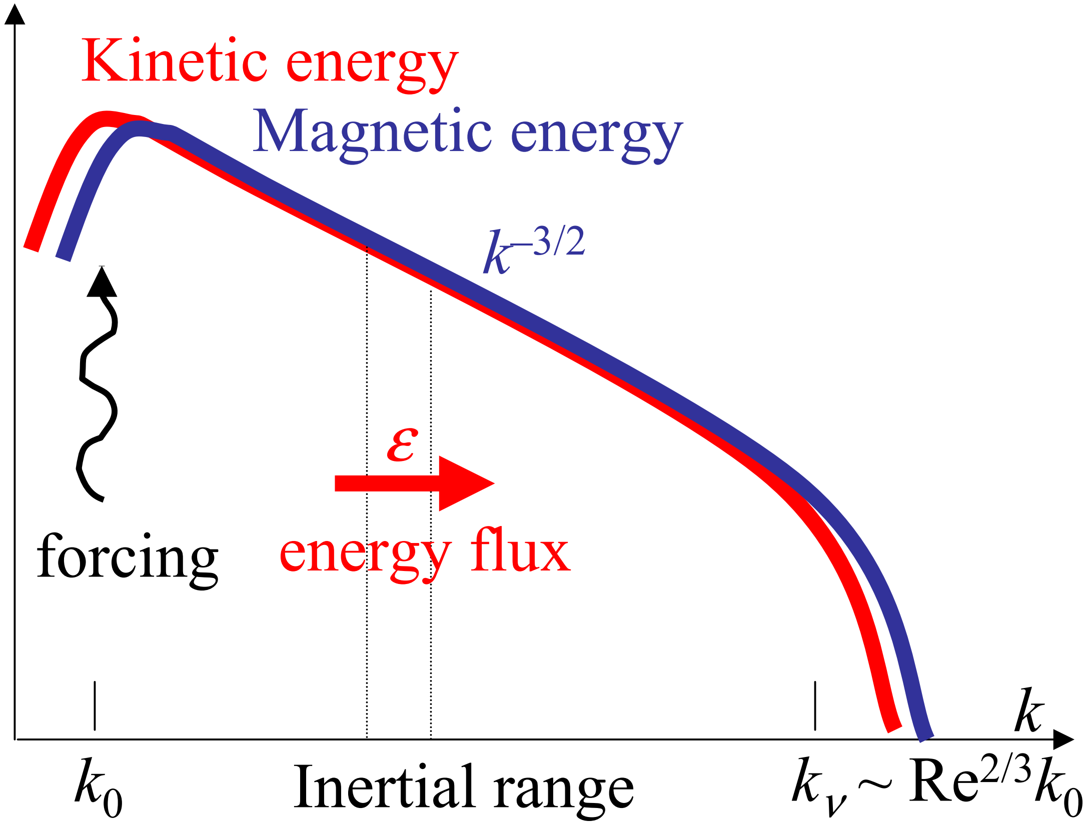
- **Very important: they mediate interaction between Alfvén waves**
  (in the weak-interaction limit, Alfvén waves are passive
  with respect to $k_{||} = 0$ modes)

- **They are not Alfvén waves themselves, rather, they are 2D MHD:**
  liable to form long-lived low-$k_\perp$ structures?
  some evidence of $k^{-3/2}$ spectrum?
  [2D MHD: Kinney *et al.* 1995, *PoP* **2**, 3623; Biskamp & Swartz 2001, *PoP* **8**, 3282]
  [3D RMHD: Kinney & McWilliams 1998, *PRE* **57**, 7111]

- **In simulations with strong B_0, do these modes get mixed up with the Alfvénic spectrum?**
  (I think this is certainly true in Müller *et al.* simulations)

- **A numerical effect only?**
  (existence of such modes depends on periodic boundaries)

Iroshnikov-Kraichnan Turbulence



- **Strong mean field B_0**
- **Alfvénic state: $u_l \sim B_l$**
- Scale invariance
- Locality in k space

$$\varepsilon \sim u_l^2\, \tau_l^{-1} = \mathrm{const}$$

Energy at scale l

Cascade time (rate of transfer)

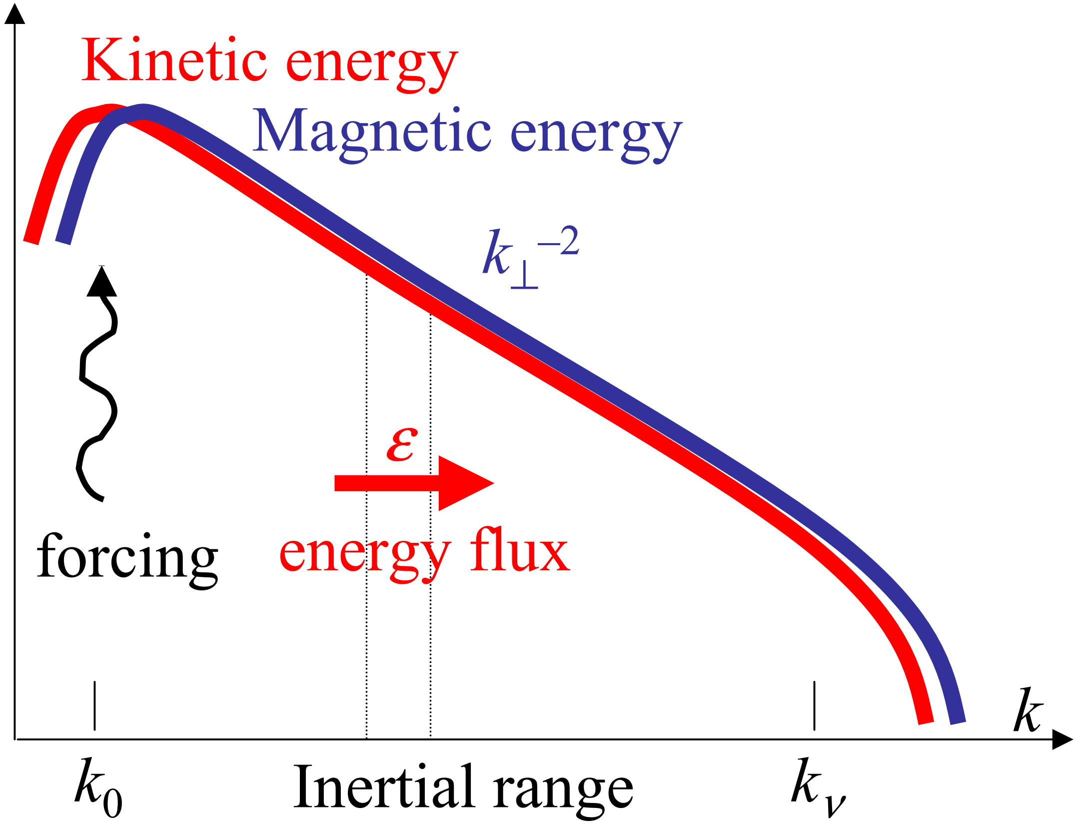
Additional physical assumptions:

- **weak interactions: $\tau_{\mathrm{eddy}} >> \tau_{\mathrm{A}}$**

$$\tau_l \sim \frac{\tau_{\mathrm{eddy}}^2}{\tau_{\mathrm{A}}} \sim \frac{l_\perp^2}{u_l^2}\frac{v_{\mathrm{A}}}{l_\parallel}$$

$$\boxed{u_l \sim (\varepsilon v_{\mathrm{A}})^{1/4} l_\perp^{1/2} l_\parallel^{-1/4}}$$

Weak MHD Turbulence



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- Scale invariance
- Locality in k space

$$\varepsilon \sim u_l^2\, \tau_l^{-1} = \text{const}$$

Energy at scale l

Cascade time (rate of transfer)

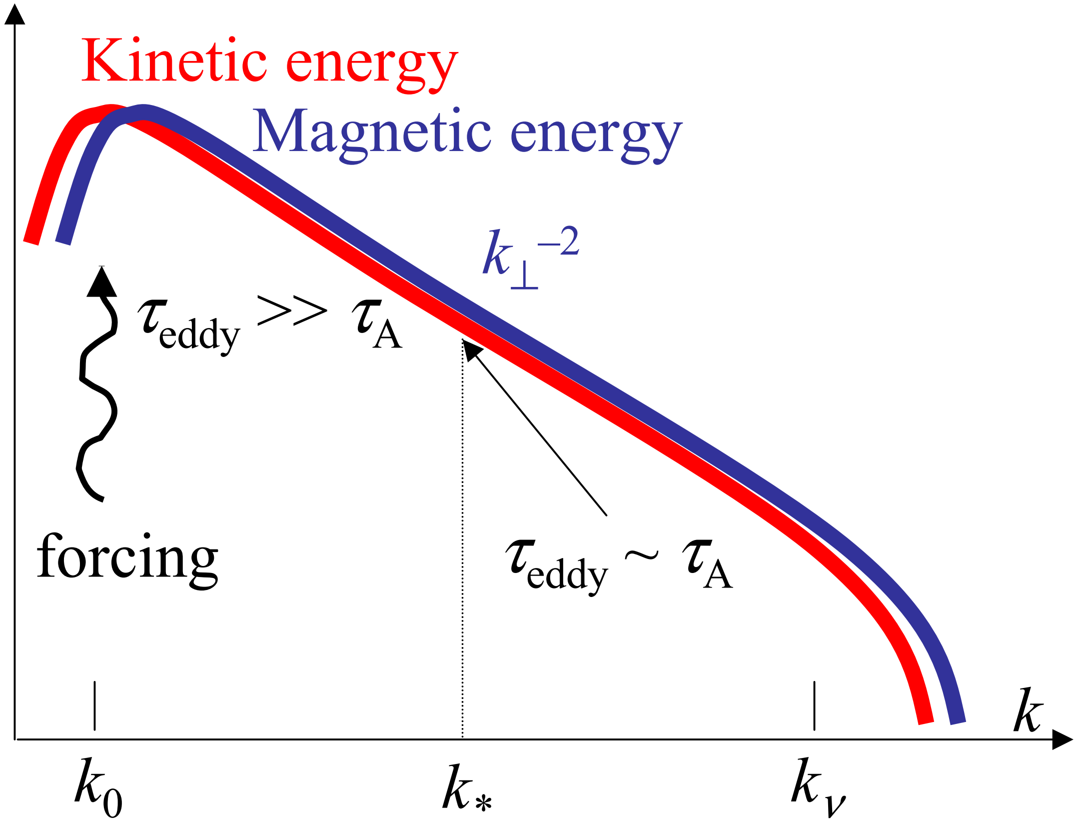
Additional physical assumptions:

- **weak interactions: $\tau_{\text{eddy}} >> \tau_{\text{A}}$**
- **extreme anisotropy: $l_{||} \sim l_0$**
  (no cascade in $k_{||}$)

$$\longrightarrow \boxed{E(k_\perp) \sim (\varepsilon k_{||} v_A)^{1/2}\, k_\perp^{-2}}$$

[e.g., Galtier *et al.* 2000, *JPP* **63**, 447; Lithwick & Goldreich 2003, *ApJ* **582**, 1220]

Weak MHD Turbulence



- **Strong mean field $\boldsymbol{B_0}$**
- **Alfvénic state: $\boldsymbol{u_l \sim B_l}$**
- Scale invariance
- Locality in k space

$$\varepsilon \sim u_l^2 \, \tau_l^{-1} = \text{const}$$

Energy at scale l

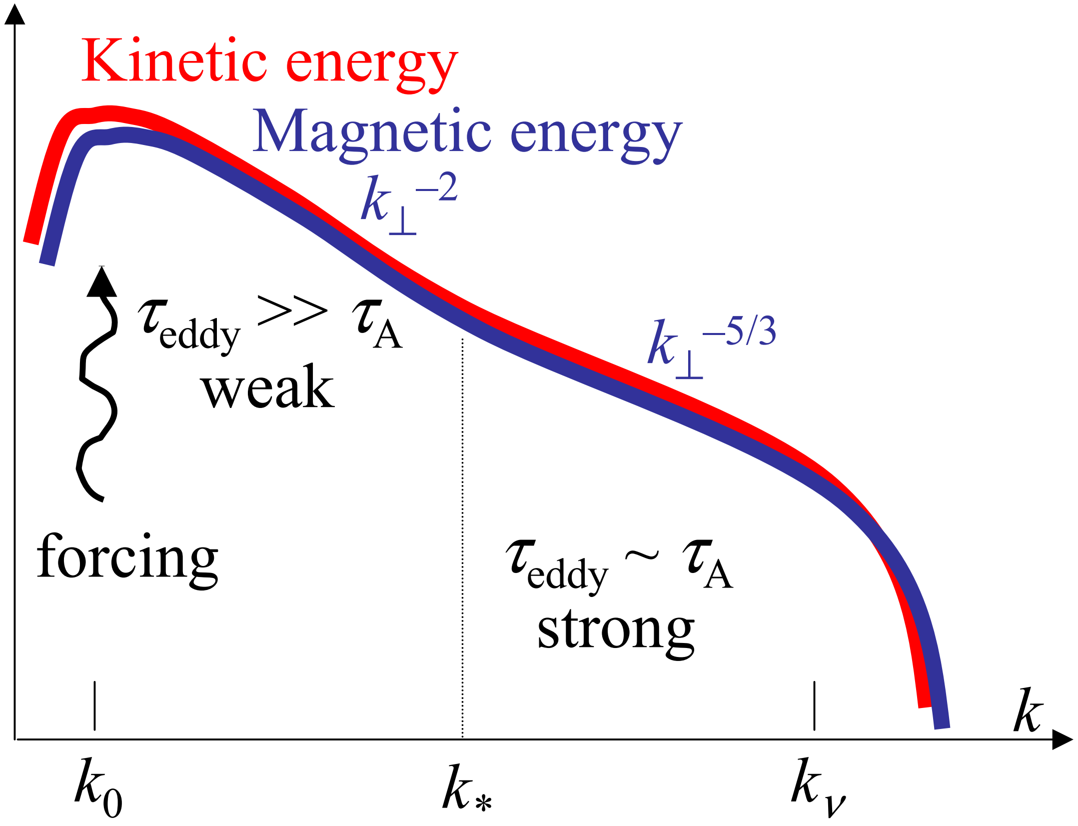
Cascade time (rate of transfer)

Weak interaction condition breaks down:

$$\frac{\tau_A}{\tau_{\text{eddy}}} \sim \frac{u_0}{v_A}\left(\frac{l_0}{l_\perp}\right)^{1/2} \sim 1 \quad \text{when} \quad l_\perp \sim l_0\left(\frac{u_0}{v_A}\right)^2 \equiv l_*$$

[e.g., Galtier *et al.* 2000, *JPP* **63**, 447; Lithwick & Goldreich 2003, *ApJ* **582**, 1220]

Goldreich-Sridhar Turbulence



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- Locality in k space

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Energy at scale l

Cascade time (rate of transfer)

Additional physical assumptions:

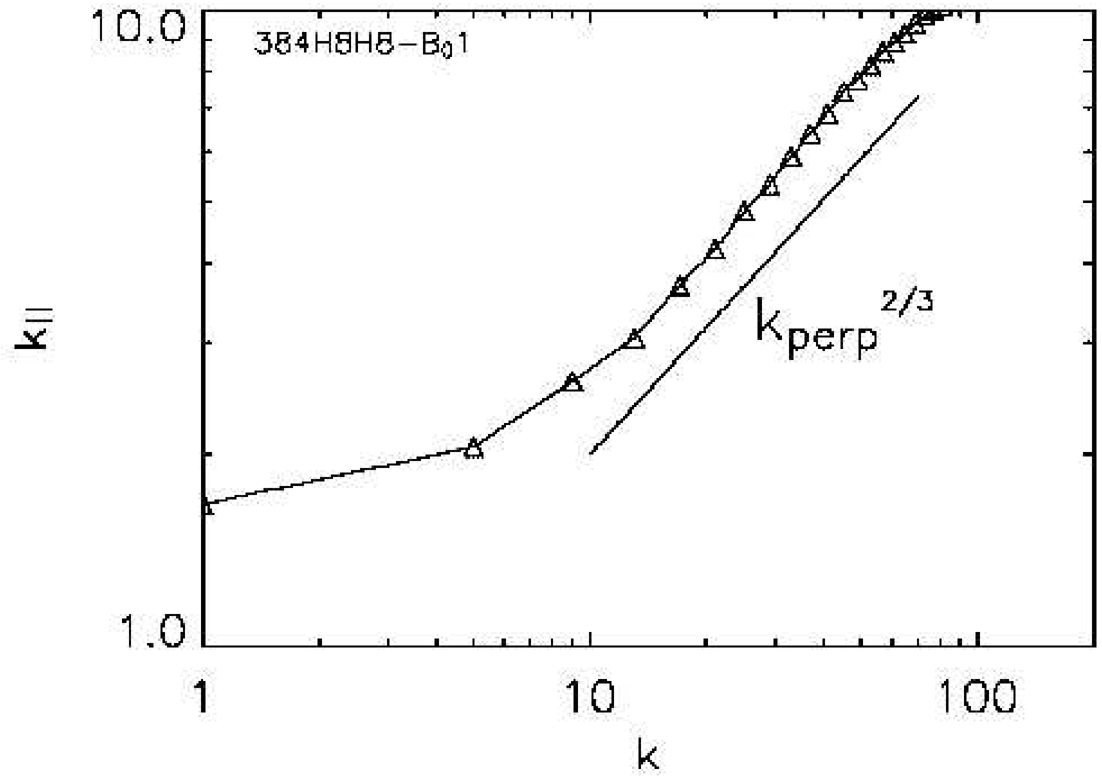
- **strong interactions: $\tau_{\rm eddy} \sim \tau_{\rm A}$** $\longrightarrow$ $\boxed{E(k_\perp) \sim \varepsilon^{2/3}\ k_\perp^{-5/3}}$ **GS95**

  (critical balance)

$$k_{||} \sim \varepsilon^{1/3} v_A^{-1}\ k_\perp^{2/3}$$

[Goldreich & Sridhar 1995, *ApJ* **438**, 763]

Goldreich-Sridhar Turbulence: DNS



$$\boxed{B_0 \sim u_{\rm rms}}$$

Cho *et al.* 2003, *ApJ* **595**, 812:

$$\boxed{l_{||} \sim l_{\perp}^{2/3}}$$

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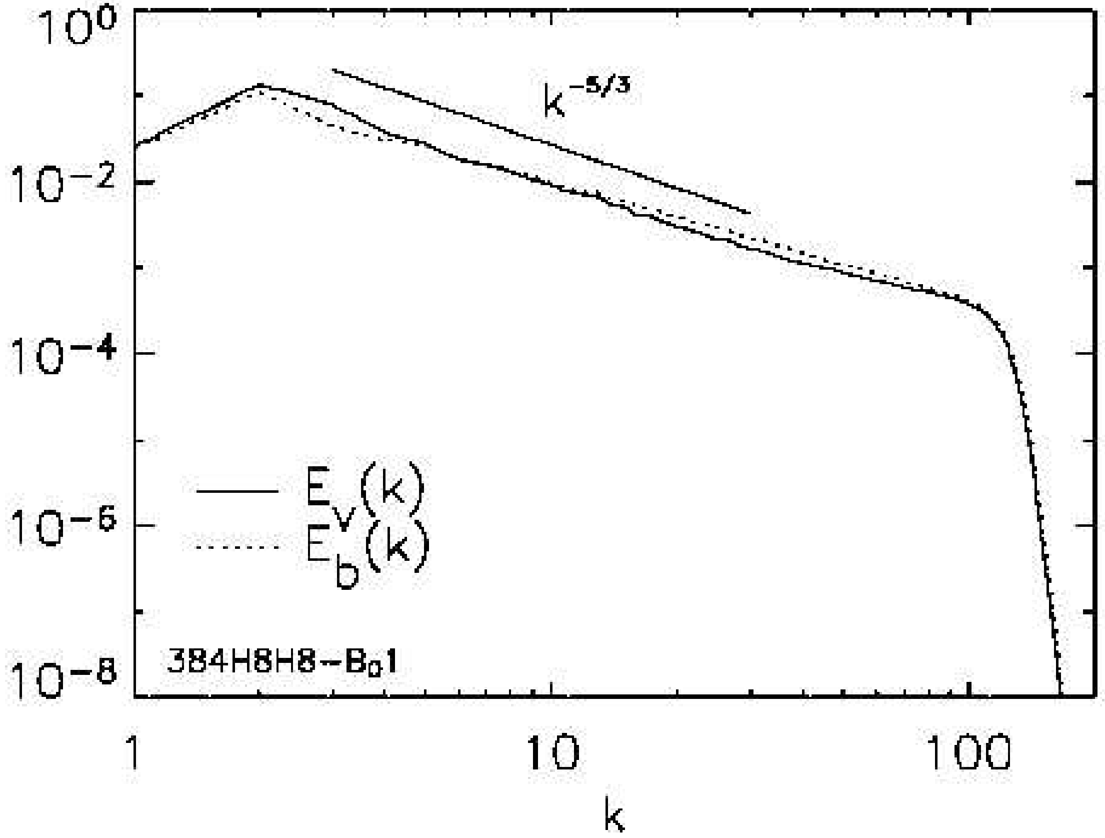

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$\boxed{E \sim k^{-5/3}}$

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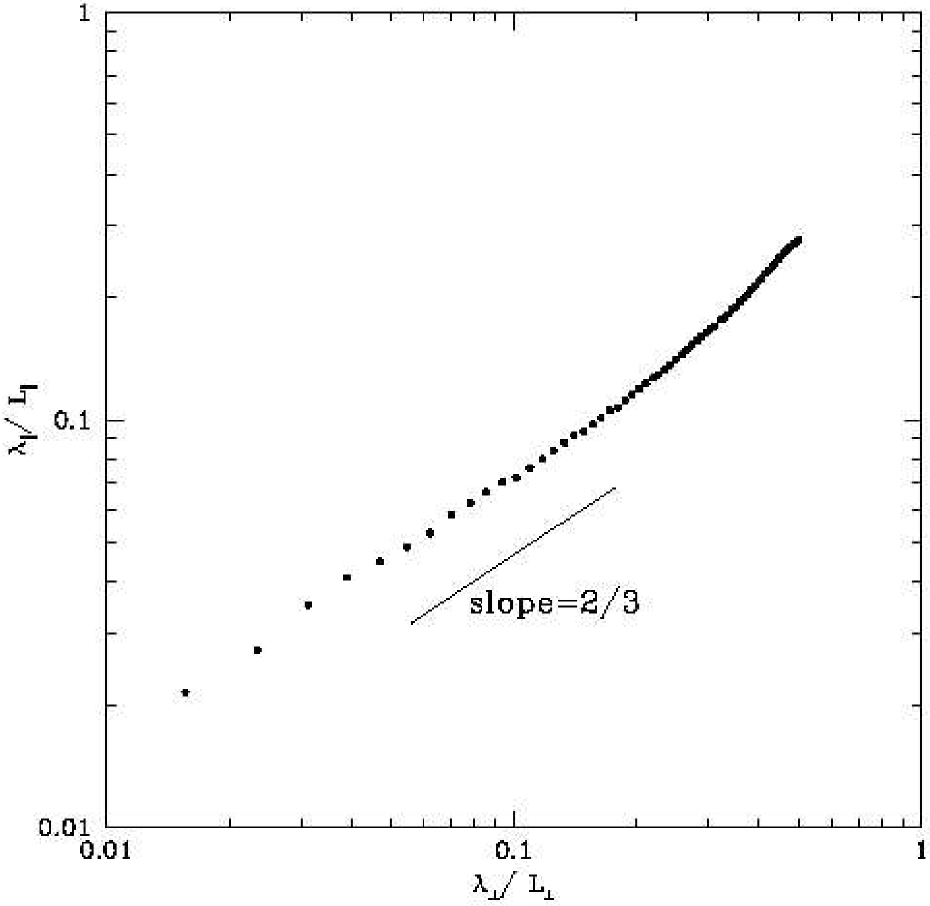

$\boxed{B_0 = 100\, u_{\rm rms}}$

Maron & Goldreich 2001, *ApJ* **554**, 1175:

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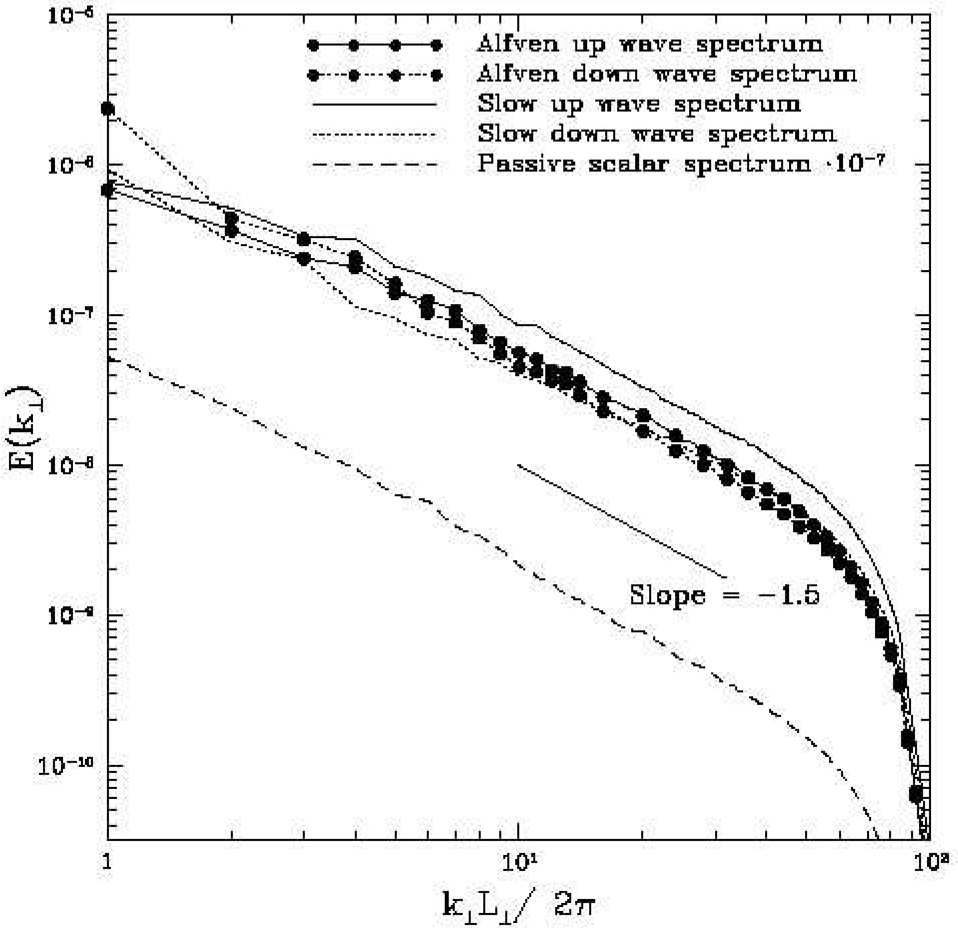

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spectra closer to $k_\perp^{-3/2}$

Spectra do not fit!

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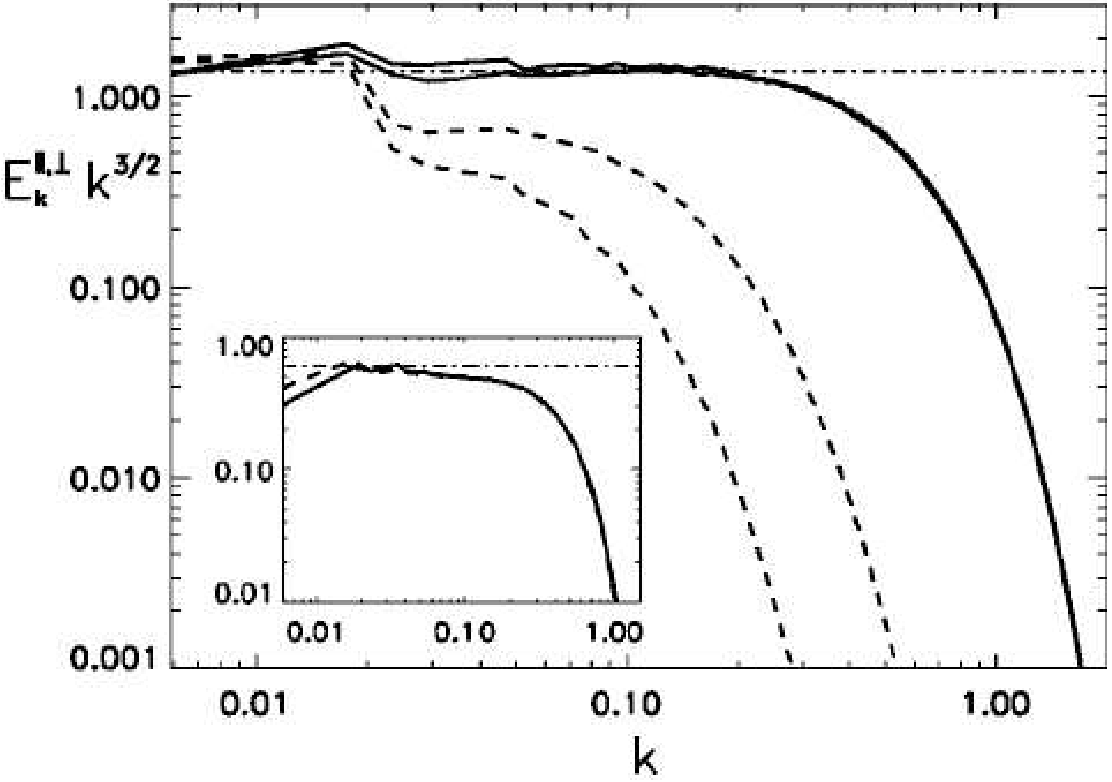

$B_0 \approx 5, 10u_{\rm rms}$

Müller *et al.* 2003, *PRE* **67**, 066302:

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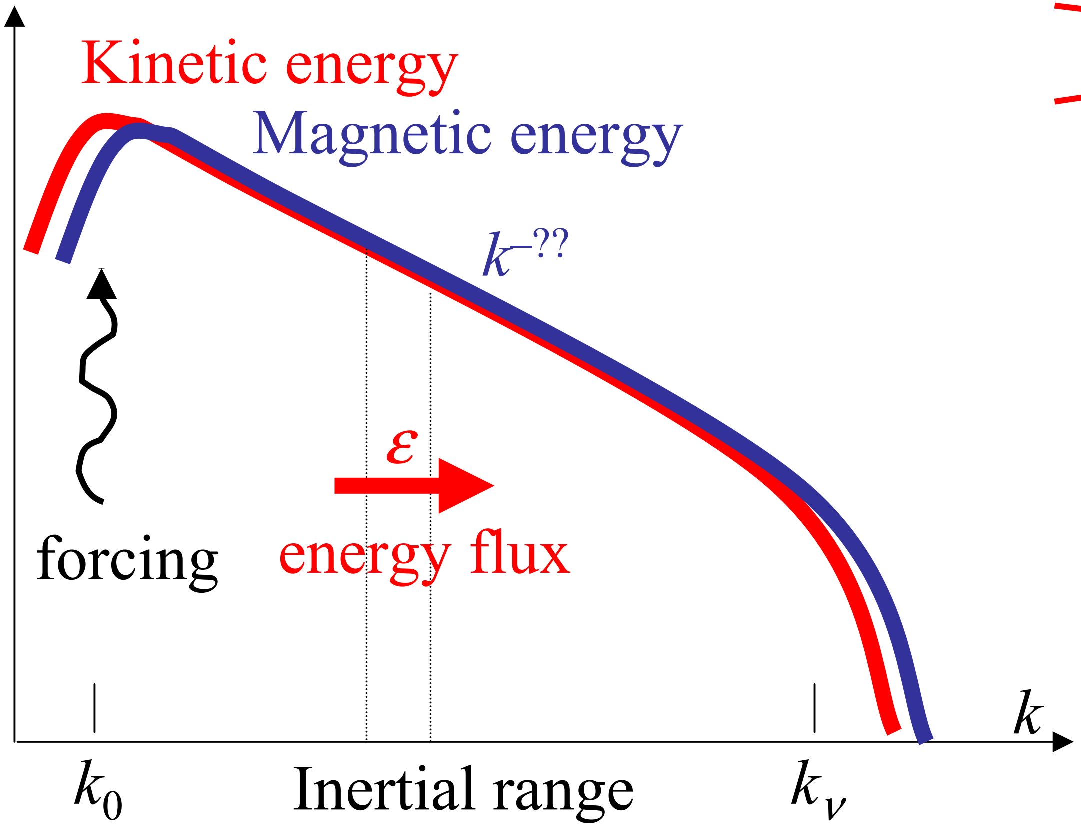
Isotropic MHD Turbulence: No Mean Field



- ~~**Strong mean field B_0**~~
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- Locality in k space

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Energy at scale l

Cascade time (rate of transfer)

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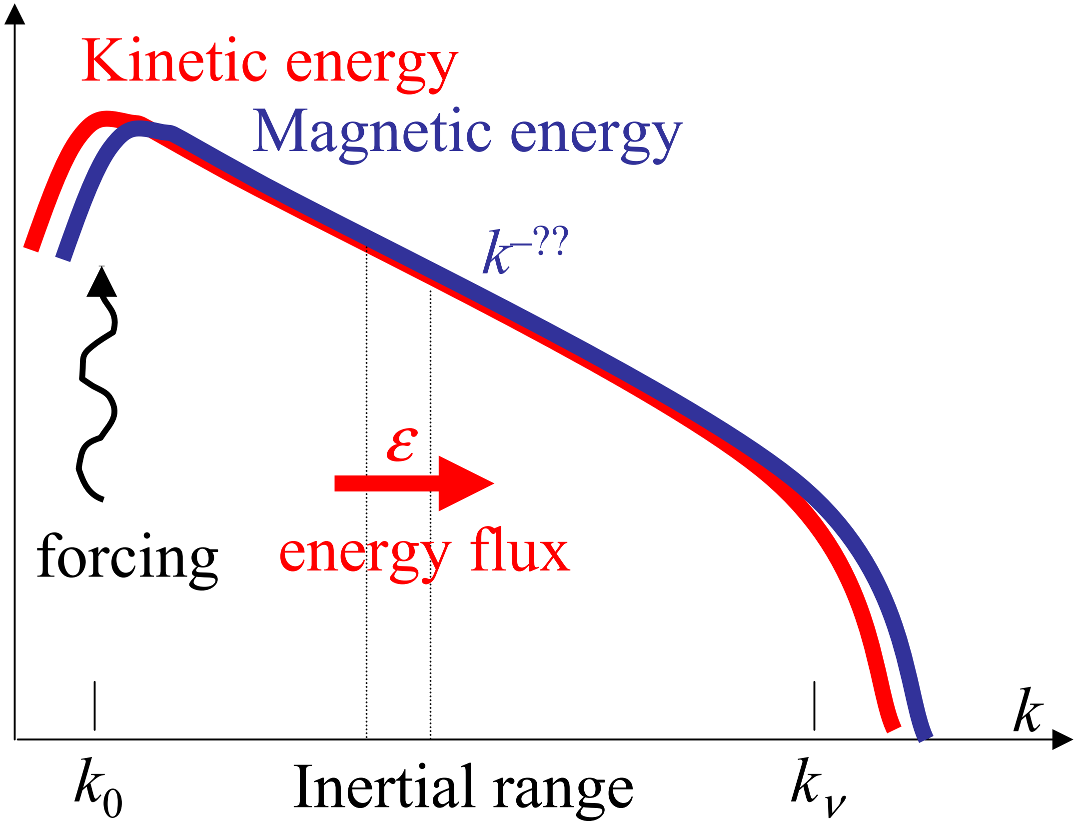

- **Strong *large-scale* field?**
- **Alfvénic state: $u_l \sim B_l$**
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- Locality in k space

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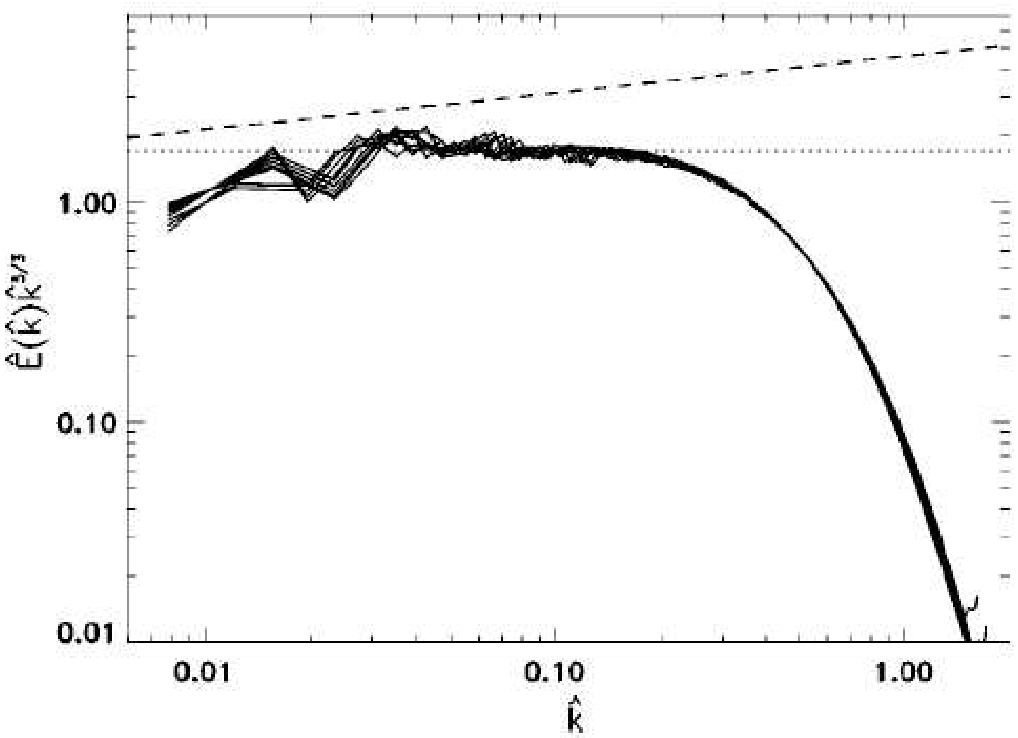
Energy at scale l

Cascade time (rate of transfer)

etc…

Isotropic MHD Turbulence: DNS (Decaying)



Biskamp & Müller 2000,
PoP **7**, 4889:

$E(k) \sim k^{-5/3}$ claimed

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Energy at
scale *l*

Cascade time
(rate of transfer)

etc…

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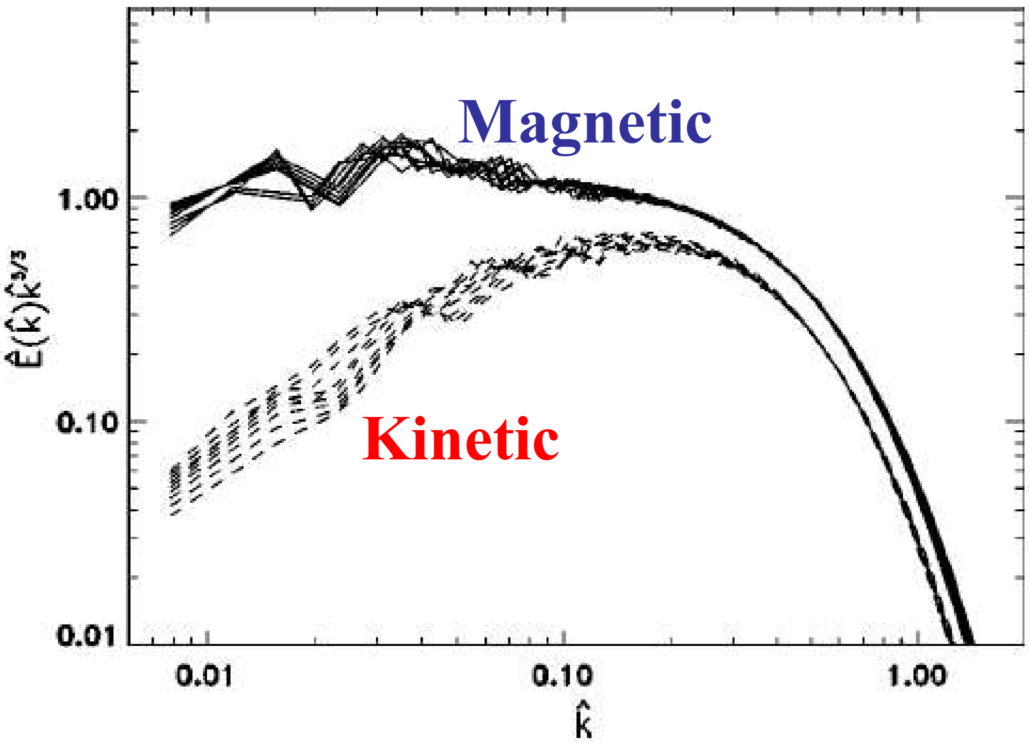

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$$\varepsilon \sim u_l^2\ \tau_l^{-1} = \text{const}$$

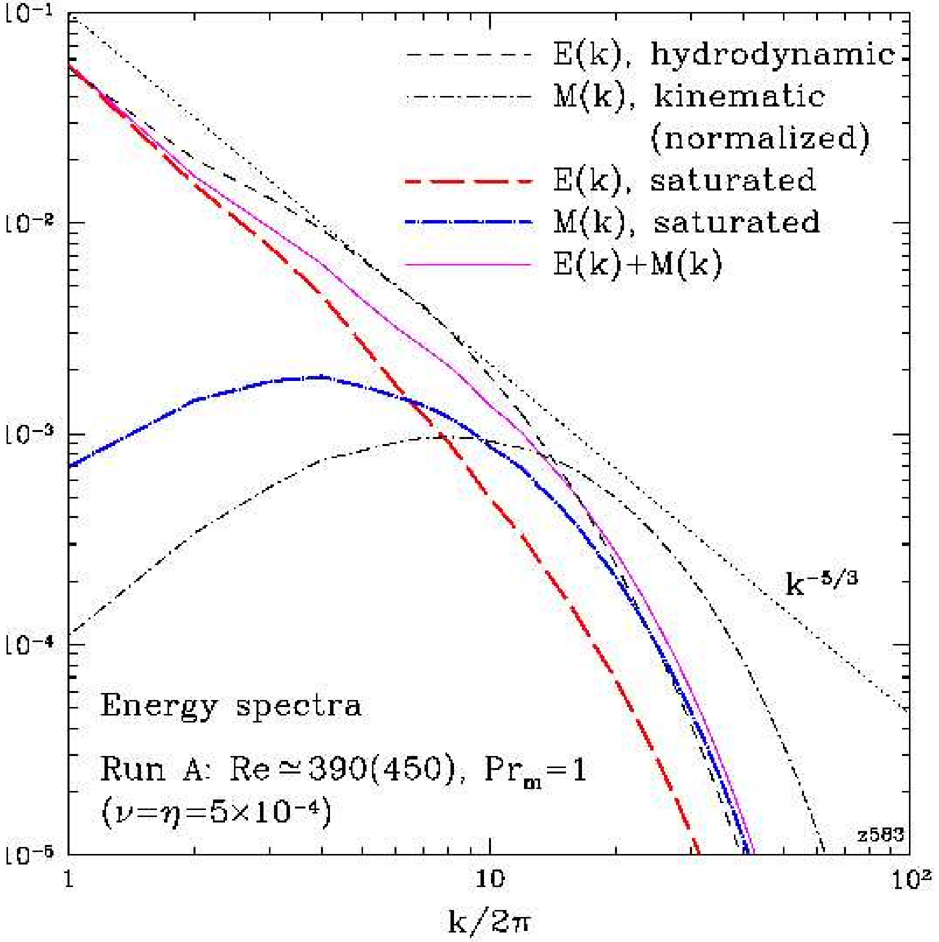
Energy at
scale l

Cascade time
(rate of transfer)

etc…

NB: decay controlled by large-scale force-free structure

Isotropic MHD Turbulence: DNS (Forced)



AAS *et al.* 2004, *ApJ* **612**, 276

See also Maron *et al.* 2004, *ApJ* **603**, 569

Haugen *et al.* 2004, *PRE* **70**, 016308

- ~~**Strong *large-scale* field?**~~
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Cascade time (rate of transfer)

etc…

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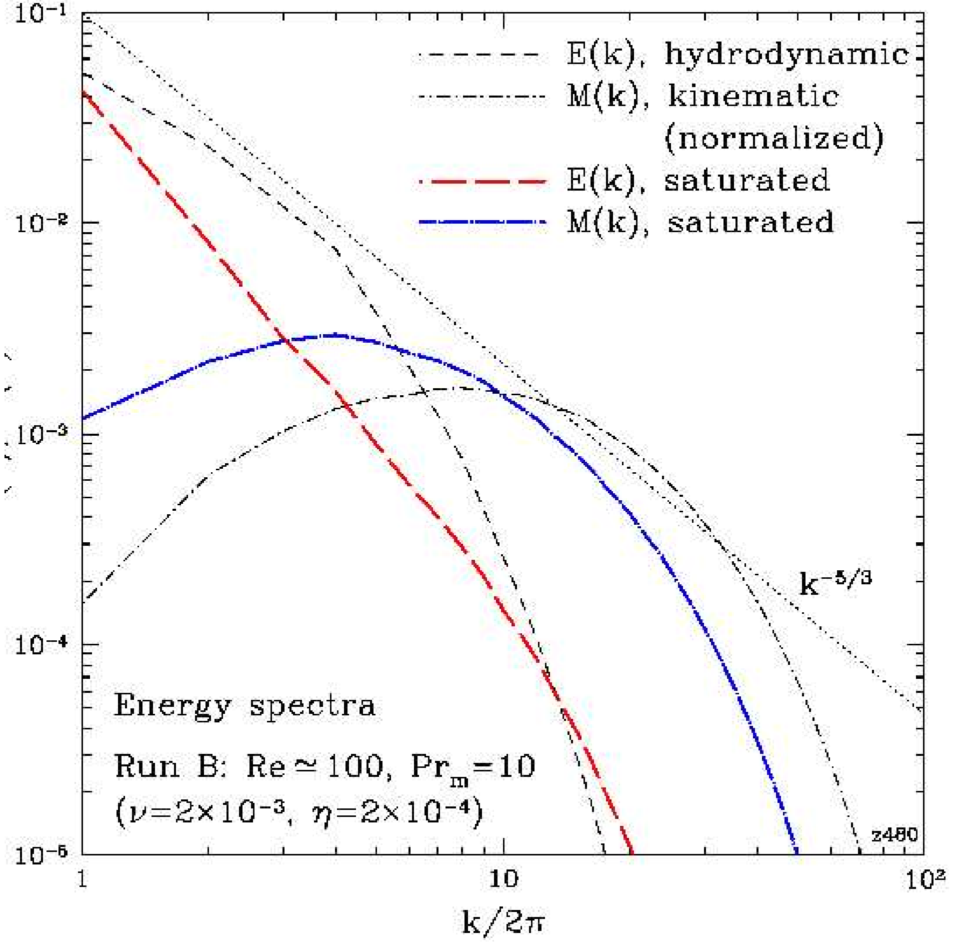

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AAS *et al.* 2004, *ApJ* **612**, 276

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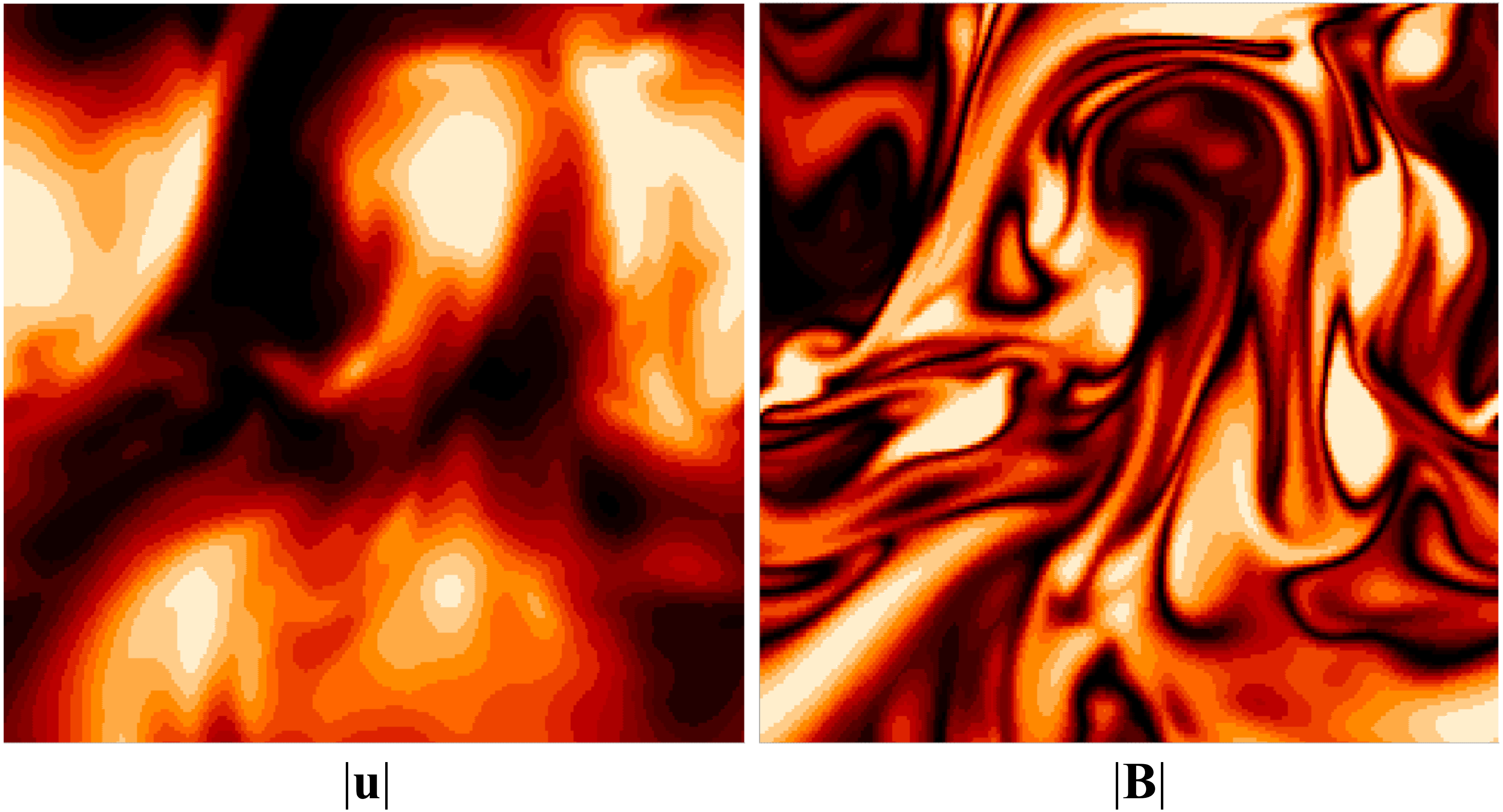
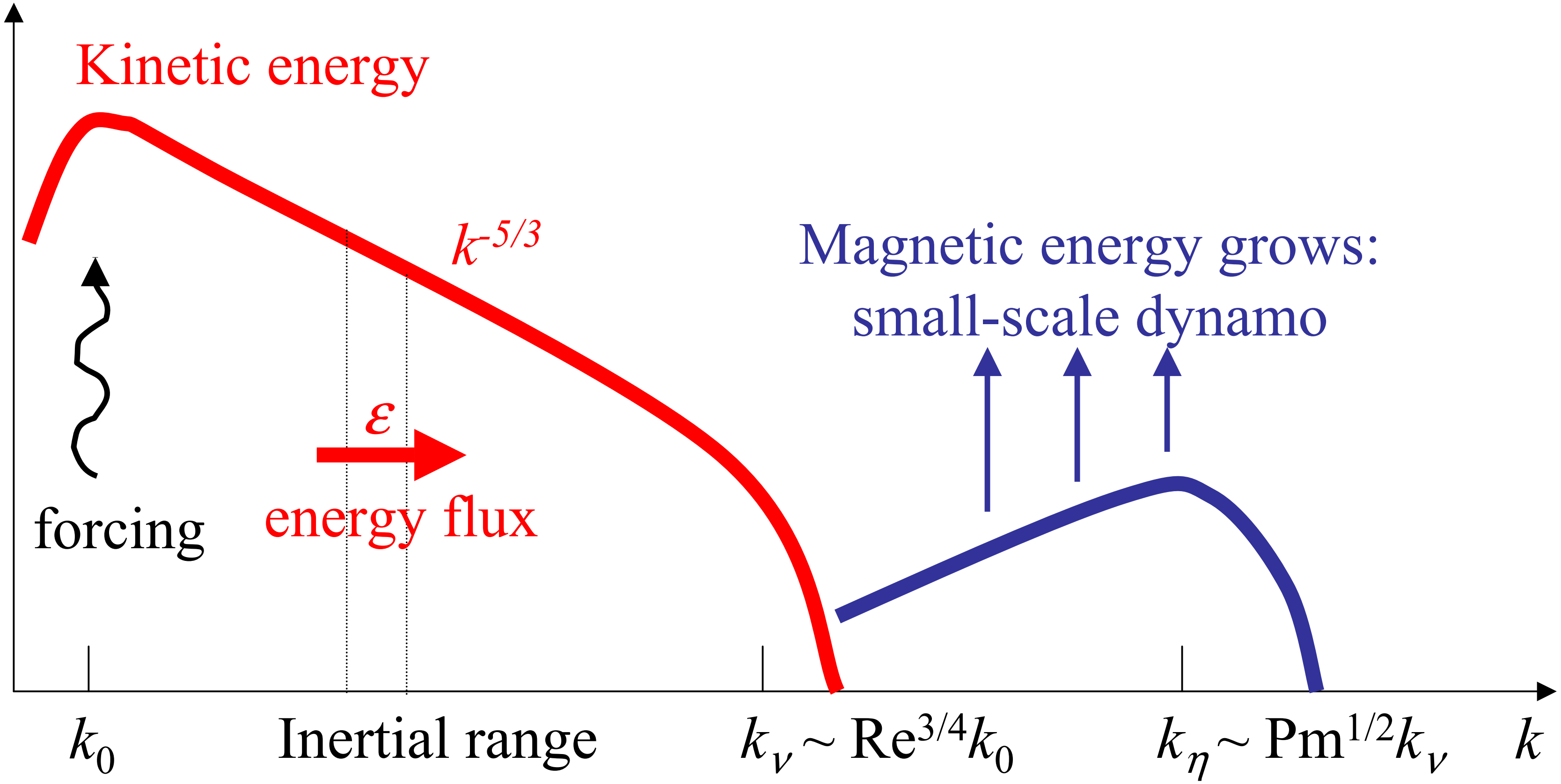


Excess of magnetic energy at small scales

[AAS *et al.* 2004, *ApJ* **612**, 276]

Two Scale Ranges in the Problem

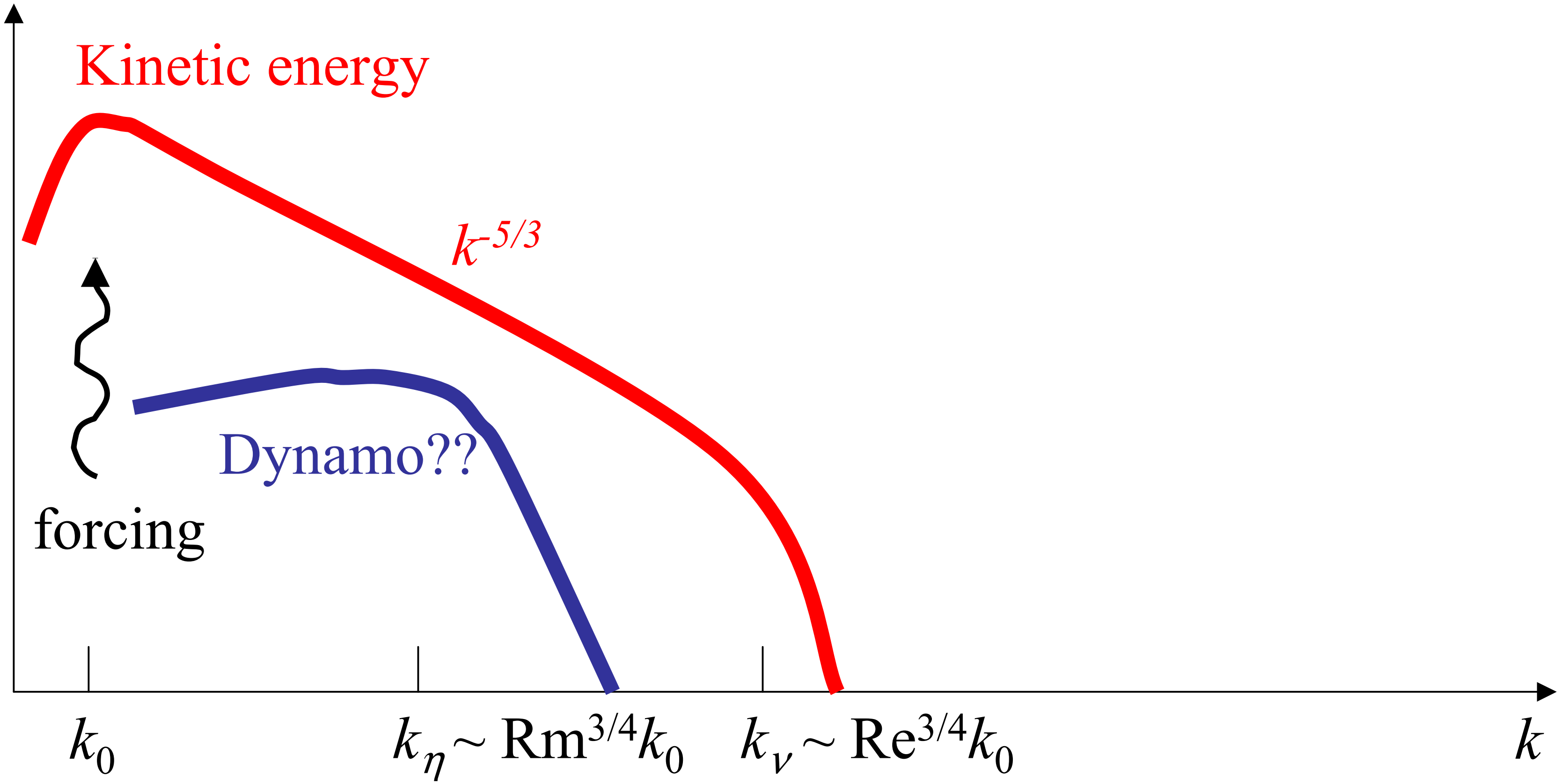


A key parameter: ***magnetic Prandtl number***

Pm = $\nu/\eta \sim 2.6\times10^{-5}\, T^4/n$ (ionised) or $1.7\times10^{7}\, T^2/n$ (with neutrals)

- **Pm >>1**: galaxies (10^{14}), clusters (10^{29})
- **Pm <<1**: planets (10^{-5}), stars ($10^{-7}\ldots10^{-4}$), protostellar discs (10^{-8}), liquid-metal laboratory dynamos (10^{-6})

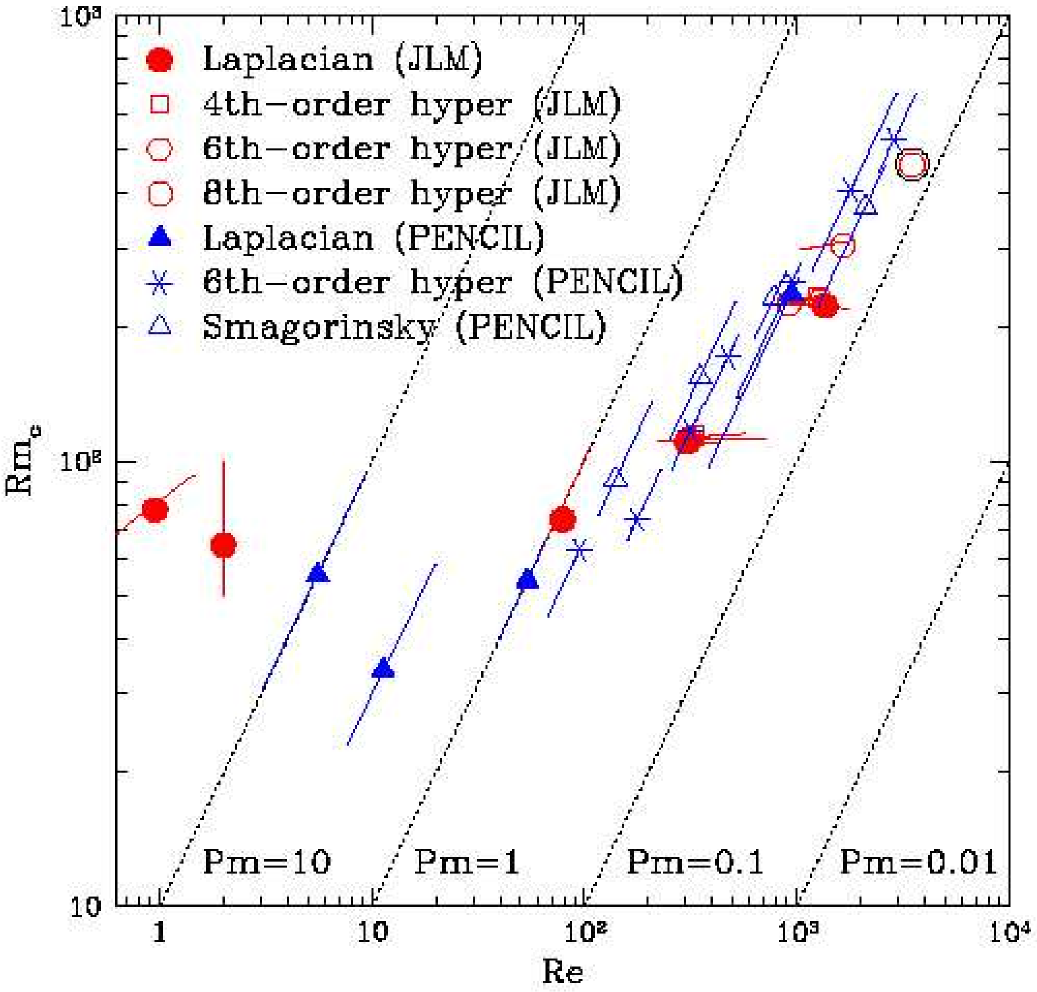
The Case of Pm << 1



Pr << 1

Re >> Rm >> 1

The Case of Pm << 1



Results from simulations with random forcing

[Schekochihin *et al.* 2004, *PRL* **92**, 054502

Haugen *et al.* 2004, *PRE* **70**, 016308

Schekochihin *et al*. 2005, *ApJ* **625**, L115]

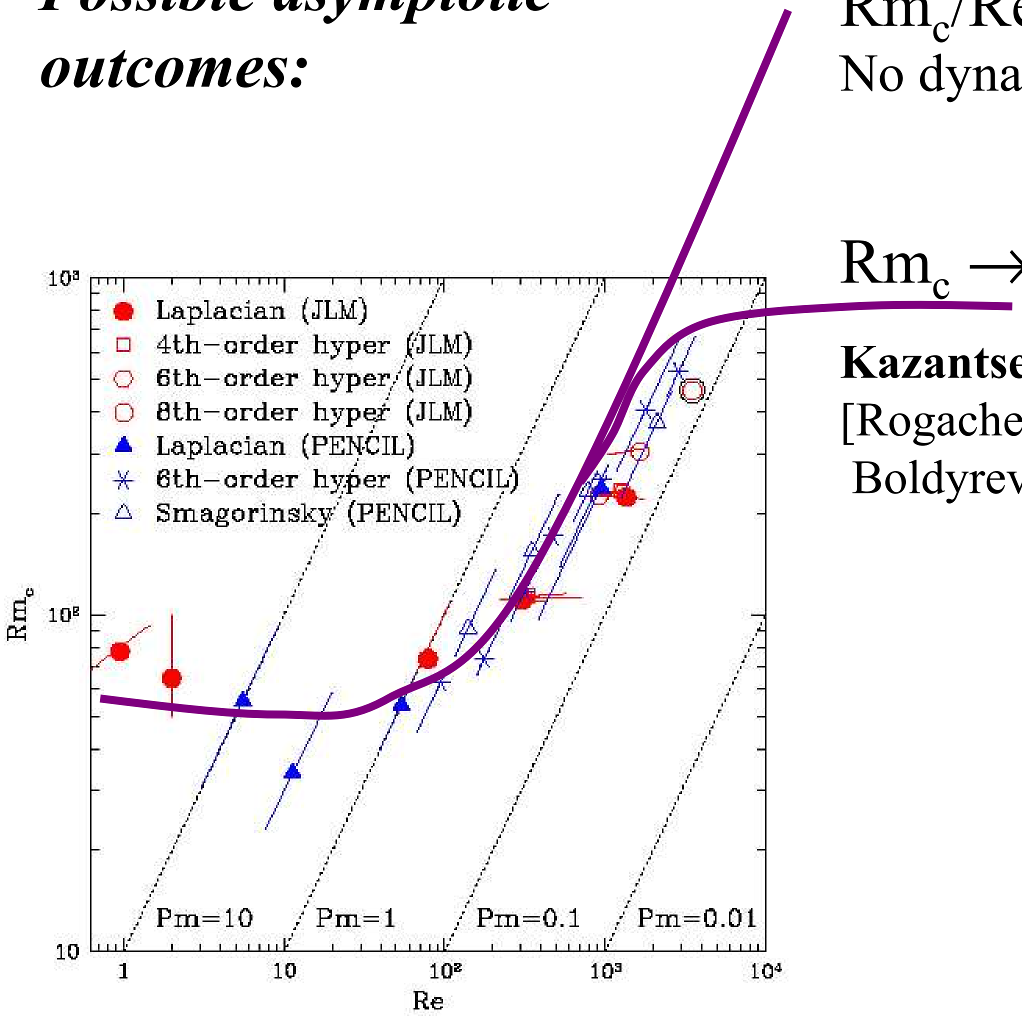
Small-scale dynamo is quite hard to get numerically in the Pm<<1 regime:

Rm_c grows with Re

$Rm_c \to$? as Re $\to \infty$ (unknown)

The Case of Pm << 1

Possible asymptotic outcomes:



$Rm_c/Re \to \text{const} = Pm_c \sim 0.1$

No dynamo at Pm << 1

OR

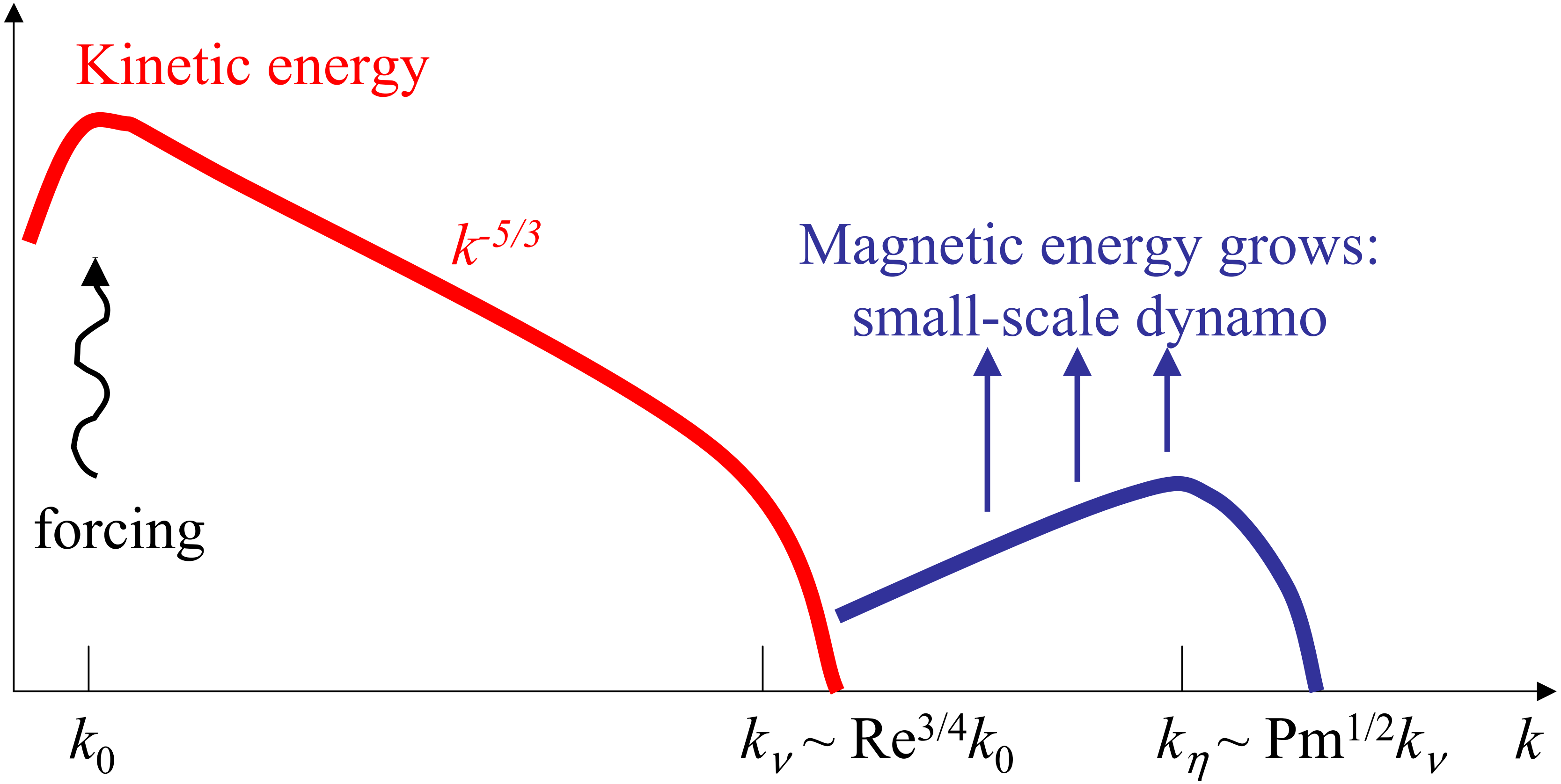
$Rm_c \to \text{const}$

Kazantsev model supports this possibility

[Rogachevskii & Kleeorin 1997, *PRE* **56**, 417

Boldyrev & Cattaneo 2004, *PRE* **92**, 144501]

Small-Scale Dynamo at Pm ≥ 1

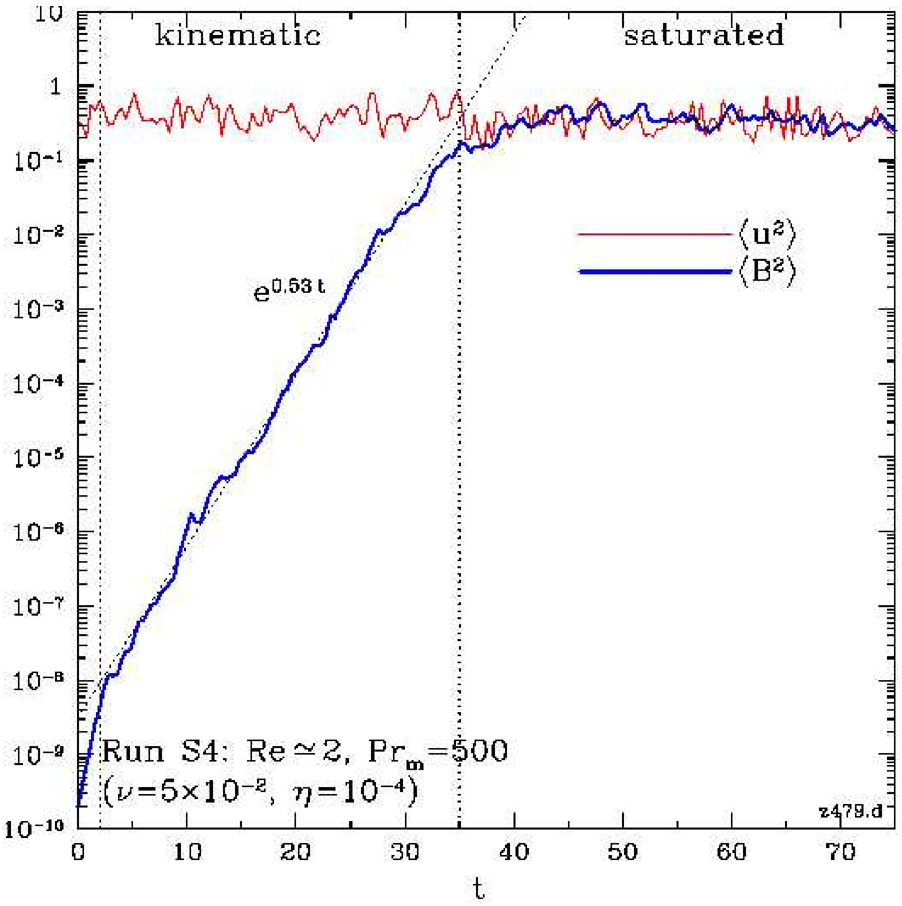


$\mathrm{Pm} \gg 1$
$\mathrm{Rm} \gg \mathrm{Re} \gg 1$

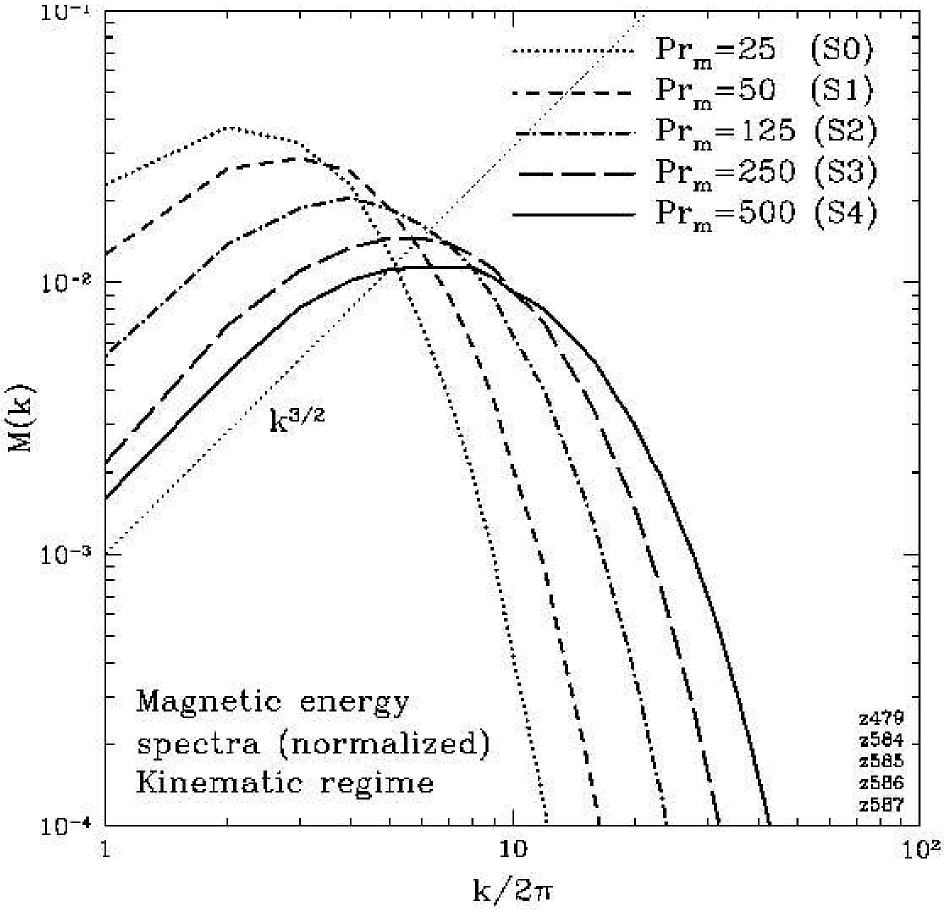
Pm ≥ 1: it is well established numerically that an initial weak seed field will grow at the smallest scales provided $\mathrm{Rm} > \mathrm{Rm}_c \sim 10^2$

[Meneguzzi, Frisch & Pouquet 1981, *PRL* **47**, 1060]

Small-Scale Dynamo: DNS



- $\langle B^2\rangle$ grows exponentially, then saturates



- Field at the resistive scale ($k_\eta \sim \mathrm{Pm}^{1/2} k_\nu$)

Quantitative analytical theory possible…

[AAS *et al*. 2004, *ApJ* **612**, 276 and references therein]

Modeling the Flow: Kazantsev Model

A solvable model of a random-in-time flow: **Gaussian white noise**

$$\langle u^i(t,\mathbf{x})u^j(t',\mathbf{x}')\rangle = \delta(t-t')\kappa^{ij}(\mathbf{x}-\mathbf{x}')$$

[Kazantsev 1967, *JETP* **26**, 1031; Kraichnan 1968, *Phys. Fluids* **11**, 945]

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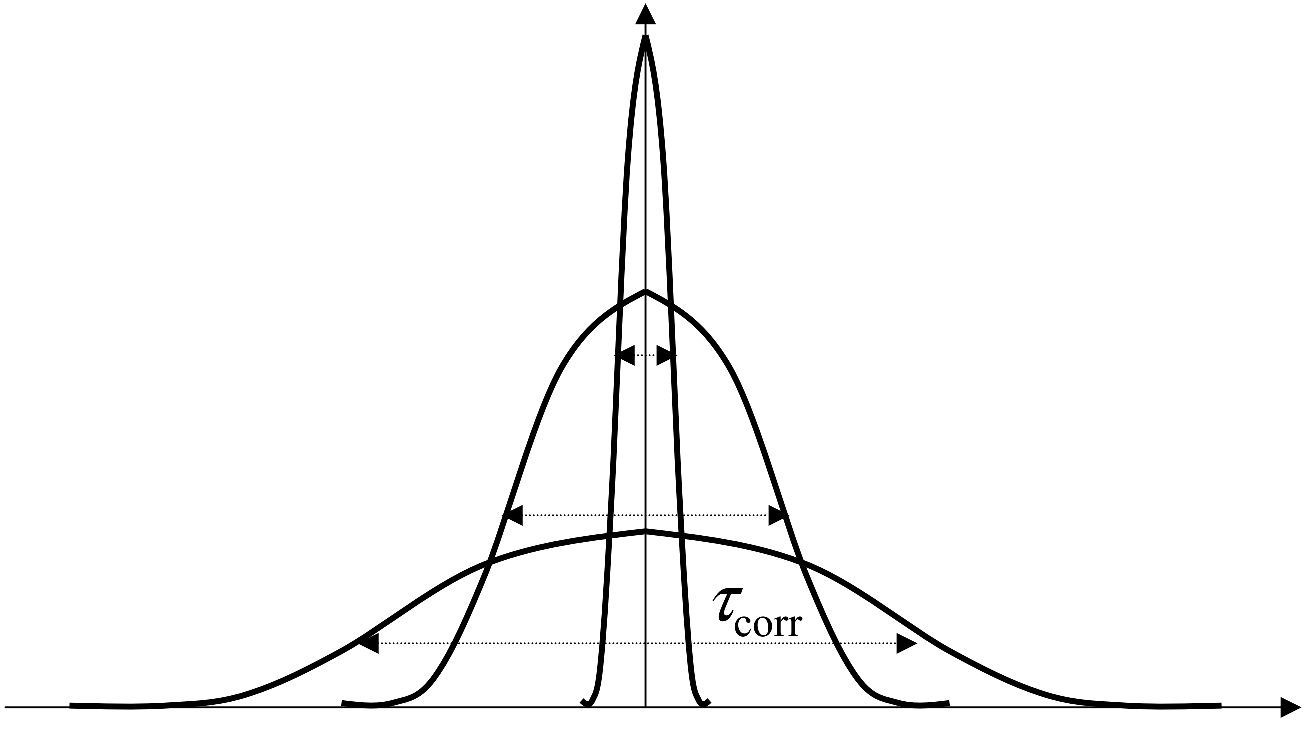
NB: Flow of infinite energy acting for time 0 ⇒ finite effect

(…a physical limit of $\tau_{\text{corr}} \to \infty$)



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Pr >> 1 ⇒ **smooth velocity at subviscous scales** *("Batchelor regime")*

$$\kappa^{ij}(\mathbf{y}) = \kappa_0\delta^{ij} - \frac{1}{2}\kappa_2\left(y^2\delta^{ij} - \frac{1}{2}y^iy^j\right) + \frac{1}{4!}\kappa_4 y^2\left(y^2\delta^{ij} - \frac{2}{3}y^iy^j\right) + \cdots$$

linear shear "bending" motion

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NOTE: Suppose we use **white forcing** in Navier-Stokes equation,

$$\langle f^i(t,\mathbf{x})f^j(t',\mathbf{x}')\rangle = \delta(t-t')\,\epsilon^{ij}(\mathbf{x}-\mathbf{x}')$$

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- Then, for Re << 1, $\langle u^i(t,\mathbf{k})\,u^j(t',\mathbf{k}')\rangle = (2\pi)^3\delta(\mathbf{k}+\mathbf{k}')\,\frac{1}{2\nu k^2}\,e^{-\nu k^2|t-t'|}\epsilon^{ij}(\mathbf{k})$

and **we connect to Kazantsev via** $\nu \to \infty,\ \epsilon^{ij}(\mathbf{k}) = \nu^2k^4\kappa^{ij}(\mathbf{k})$

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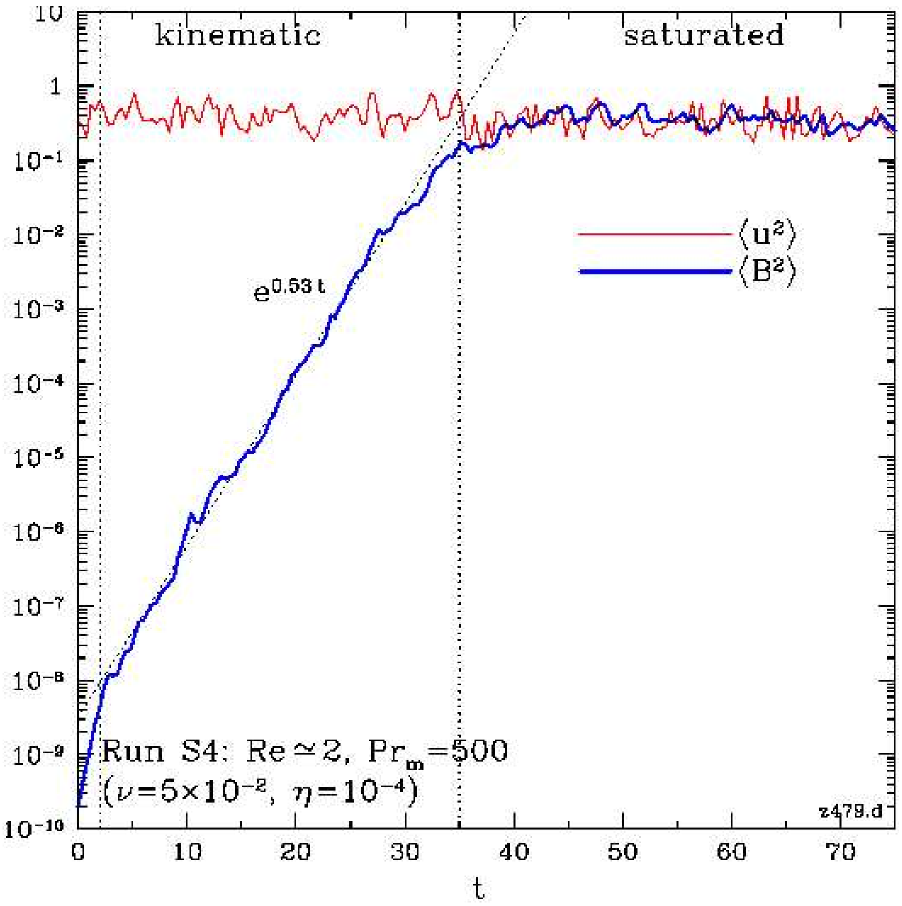
Note: Suppose we use white forcing in Navier-Stokes equation,

$$\langle f^i(t,\mathbf{x})f^j(t',\mathbf{x}')\rangle = \delta(t-t')\,\epsilon^{ij}(\mathbf{x}-\mathbf{x}')$$

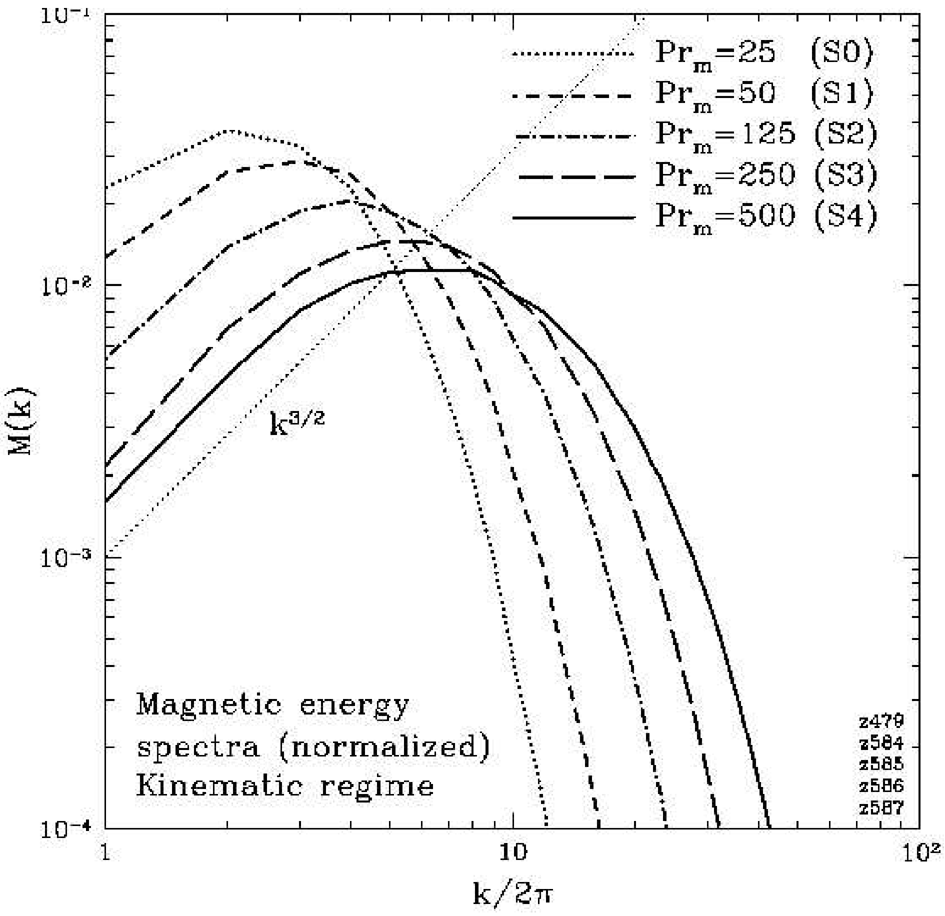
- Then, for Re << 1, $\langle u^i(t,\mathbf{k})\,u^j(t',\mathbf{k}')\rangle = (2\pi)^3\delta(\mathbf{k}+\mathbf{k}')\,\frac{1}{2\nu k^2}\,e^{-\nu k^2|t-t'|}\epsilon^{ij}(\mathbf{k})$

  and **we connect to Kazantsev via** $\nu\to\infty,\ \epsilon^{ij}(\mathbf{k}) = \nu^2k^4\kappa^{ij}(\mathbf{k})$
- **Real turbulence** when Re >> 1
- We simulate Re ~ 1 — an intermediate regime with <u>finite correlation time</u> and forcing, but no inertial range (viscous scale ~ forcing scale)

Small-Scale Dynamo: DNS



- $\langle B^2 \rangle$ **grows exponentially, then saturates**



- **Field at the resistive scale ($k_\eta \sim \mathrm{Pm}^{1/2} k_\nu$)**

[AAS *et al*. 2004, *ApJ* **612**, 276 and references therein]

Small-Scale Dynamo in the Kazantsev Model

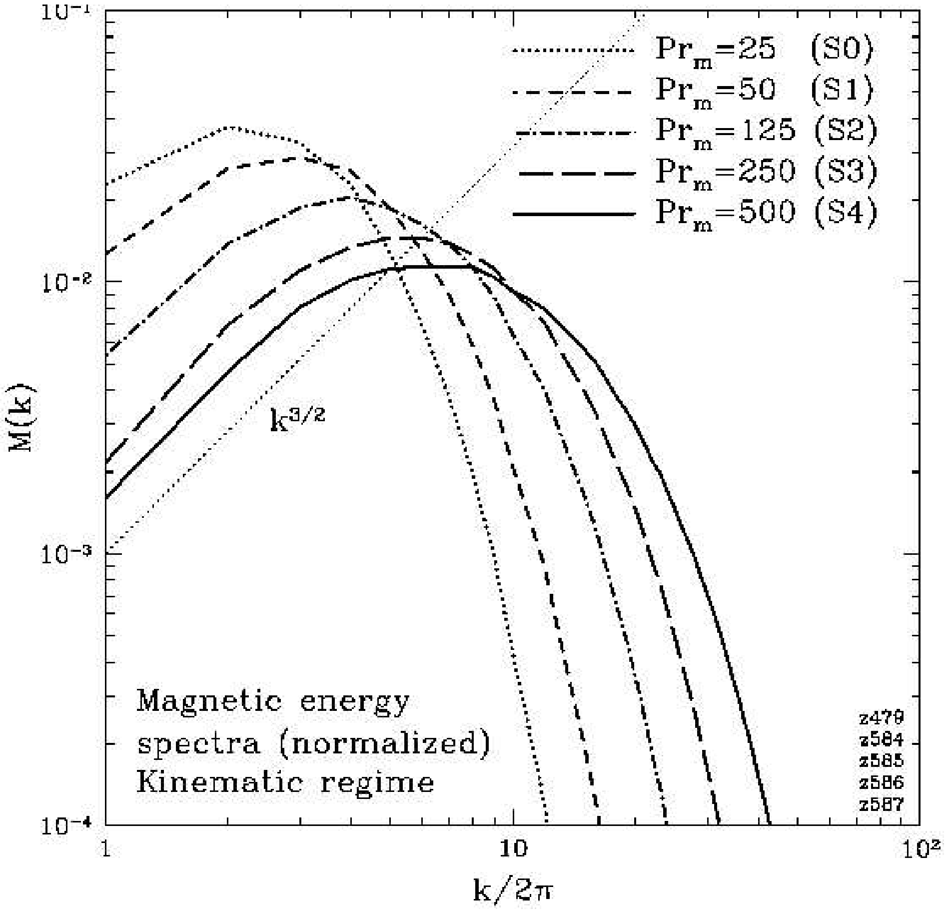
Diffusion-free regime:

$$\langle B^2\rangle \propto \exp[2\bar{\gamma}t]$$

Resistive (diffusive) regime:

$$\langle B^2\rangle \propto \exp[(3/4)\bar{\gamma}t]$$

where $\bar{\gamma} = (5/4)\kappa_2$



- $\langle B^2\rangle$ grows exponentially, then saturates

- Field at the resistive scale ($k_\eta \sim \mathrm{Pr}^{1/2} k_\nu$)

[Kulsrud & Anderson 1992, *ApJ* **396**, 606; AAS *et al.* 2002, *ApJ* **567**, 828]

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Resistive regime:

self-similar spectrum

$$M(t,k) \simeq \mathrm{const}\, e^{\lambda \bar{\gamma} t} k^{3/2} K_0(k/k_\eta)$$

where $\lambda \simeq \frac{3}{4} - \frac{\pi^2}{5[\ln(\mathrm{Pr}^{1/2})]^2}$

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Spectrum Is Not the End of Story…

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But spectra do not tell us very much about field structure!

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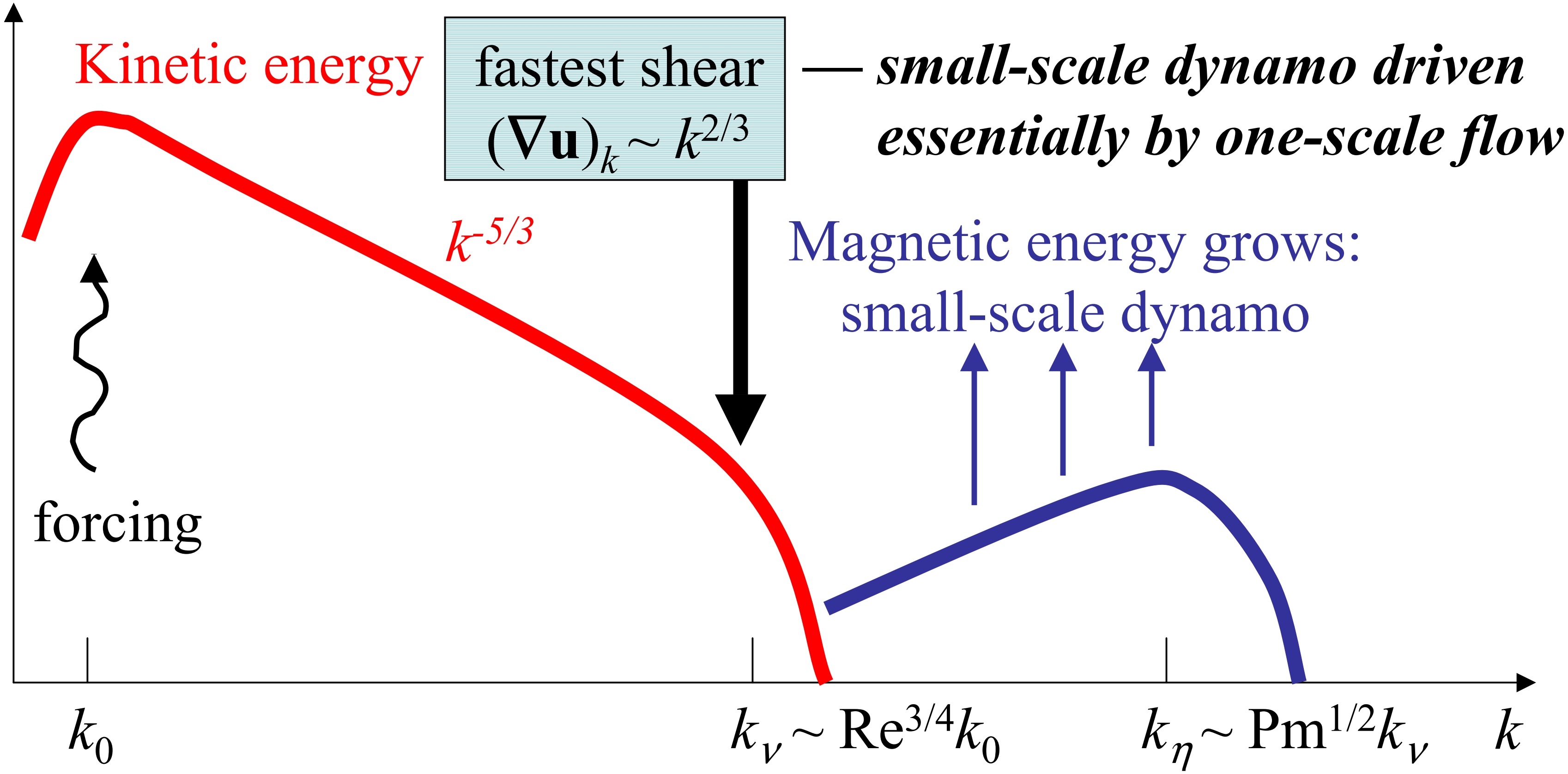
But spectra do not tell us very much about field structure!

For example, one used to argue, from the above:

back reaction $\mathbf{B}\cdot\nabla\mathbf{B} \sim k_\eta B^2 \sim \mathbf{u}\cdot\nabla\mathbf{u} \sim k_\nu u^2$ sets in when

$B^2 \sim (k_\nu / k_\eta)\, u^2 \sim \mathrm{Pr}^{-1/2} u^2 << u^2$ — **WRONG!!!**

Small-Scale Dynamo at Pm ≥ 1



Kinetic energy

fastest shear
$(\nabla \mathbf{u})_k \sim k^{2/3}$

— ***small-scale dynamo driven essentially by one-scale flow***

$k^{-5/3}$

Magnetic energy grows:
small-scale dynamo

forcing

k_0

$k_\nu \sim \mathrm{Re}^{3/4} k_0$

$k_\eta \sim \mathrm{Pm}^{1/2} k_\nu$

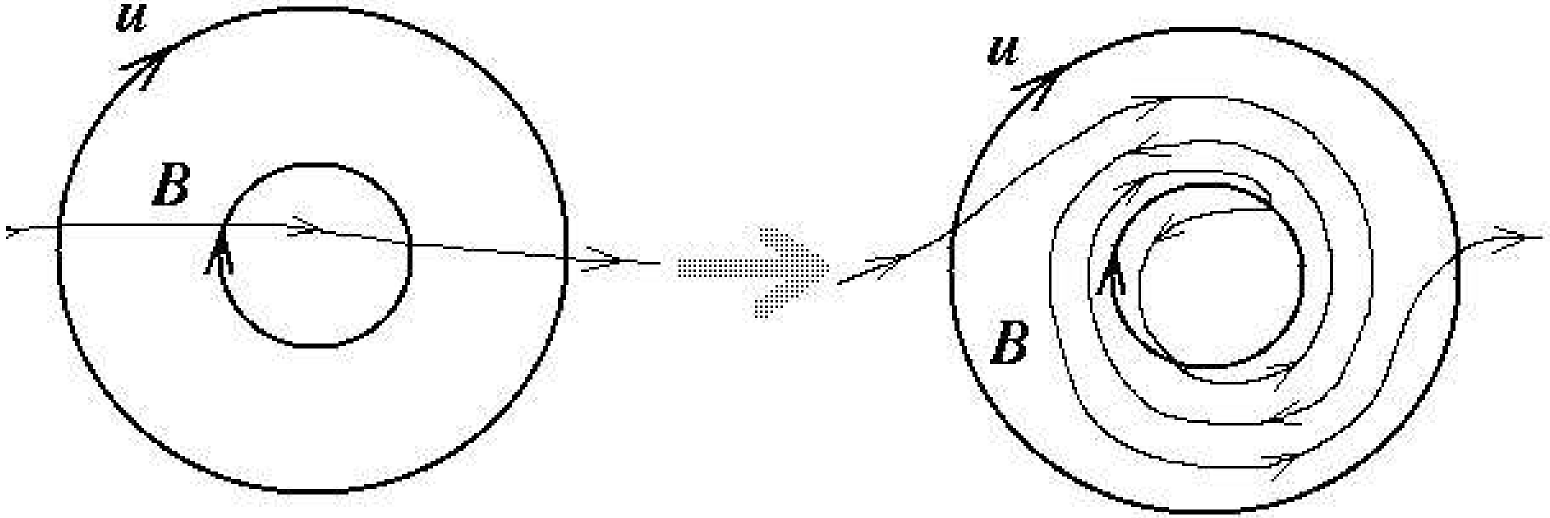
k

Pm >> 1
Rm >> Re >> 1

Dynamo mechanism: **random stretching**
Very generic!

The Folded Structure: Common Sense

The flow winds up magnetic field into ***folds****:*



Straight field

Folded field

- Direction reversals at the resistive scale, $k_\perp \sim k_\eta$
- Field varies slowly along itself: $k_{||} \sim k_{\mathrm{flow}}$

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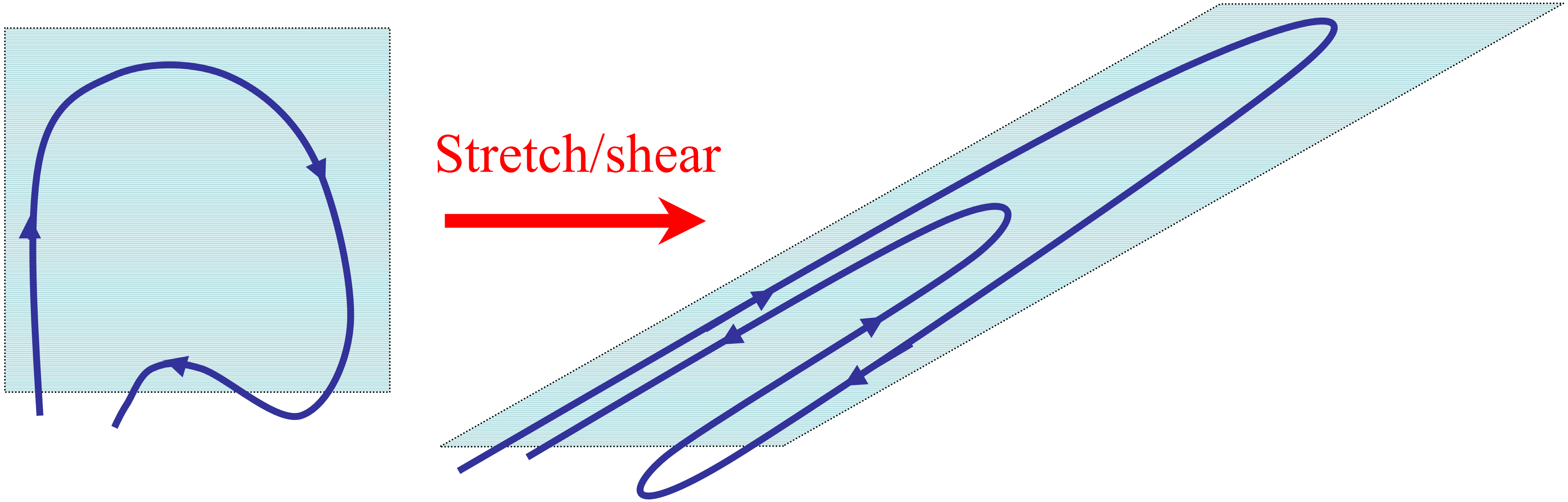

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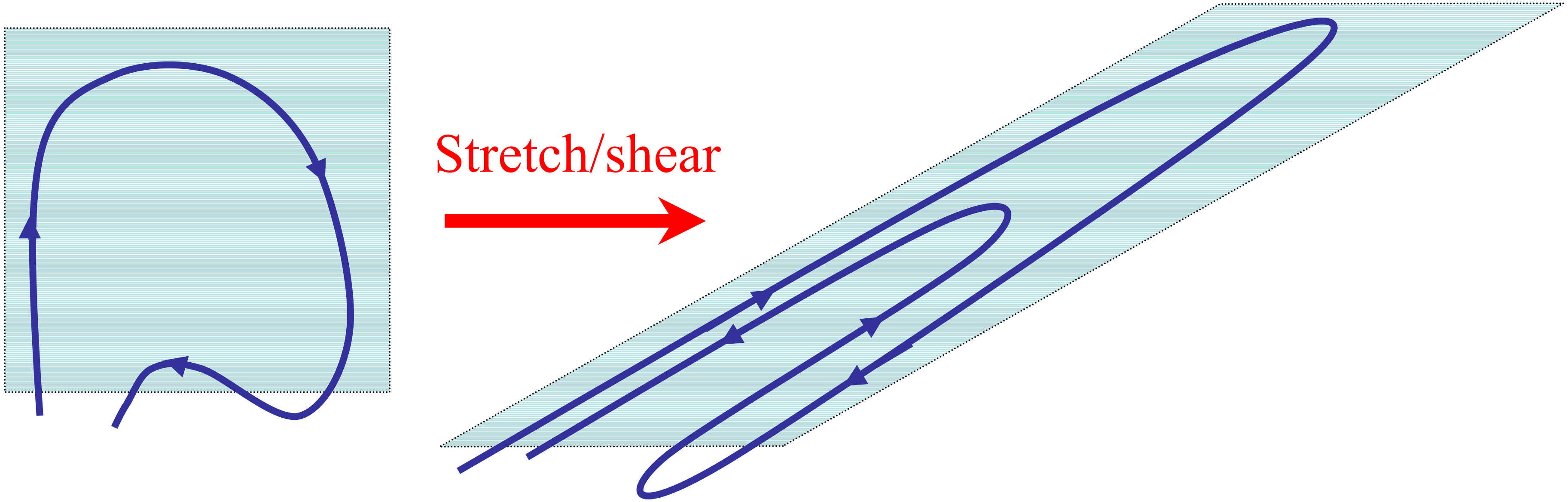


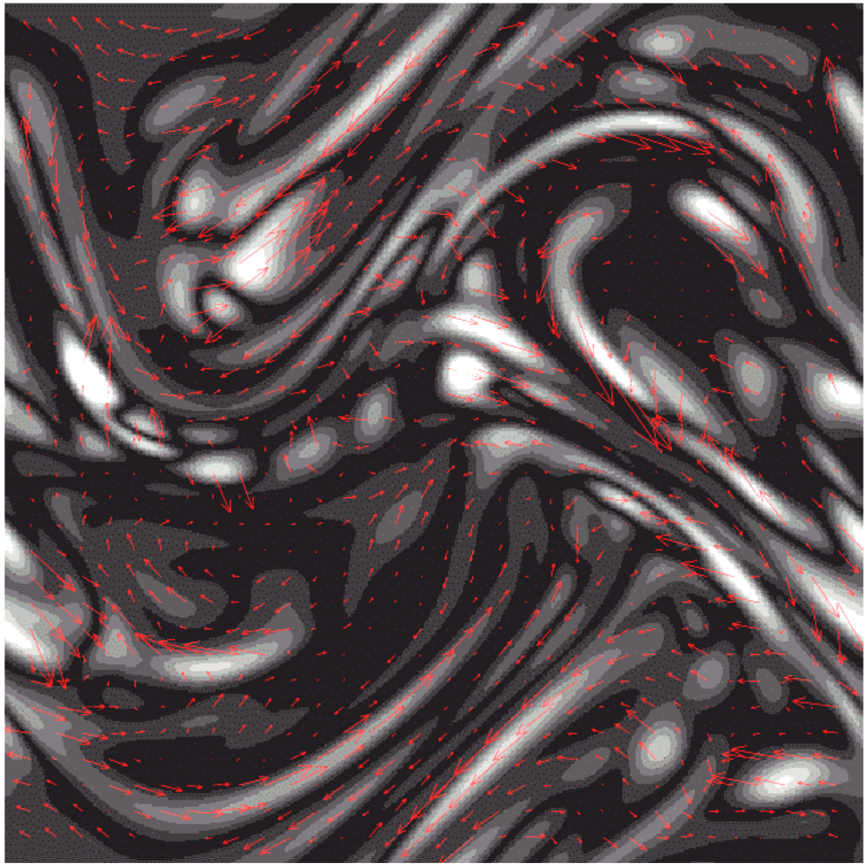
- Direction reversals at the resistive scale, $k_\perp \sim k_\eta$
- Field varies slowly along itself: $k_{||} \sim k_{\text{flow}}$

Therefore, in fact,

back reaction $\mathbf{B}\cdot\nabla\mathbf{B} \sim k_{||}B^2 \sim k_\nu B^2 \sim \mathbf{u}\cdot\nabla\mathbf{u} \sim k_\nu u^2$ sets in when

$B^2 \sim u^2$ (energy of the viscous eddies)

Folded Structure: DNS (Pm >> 1)



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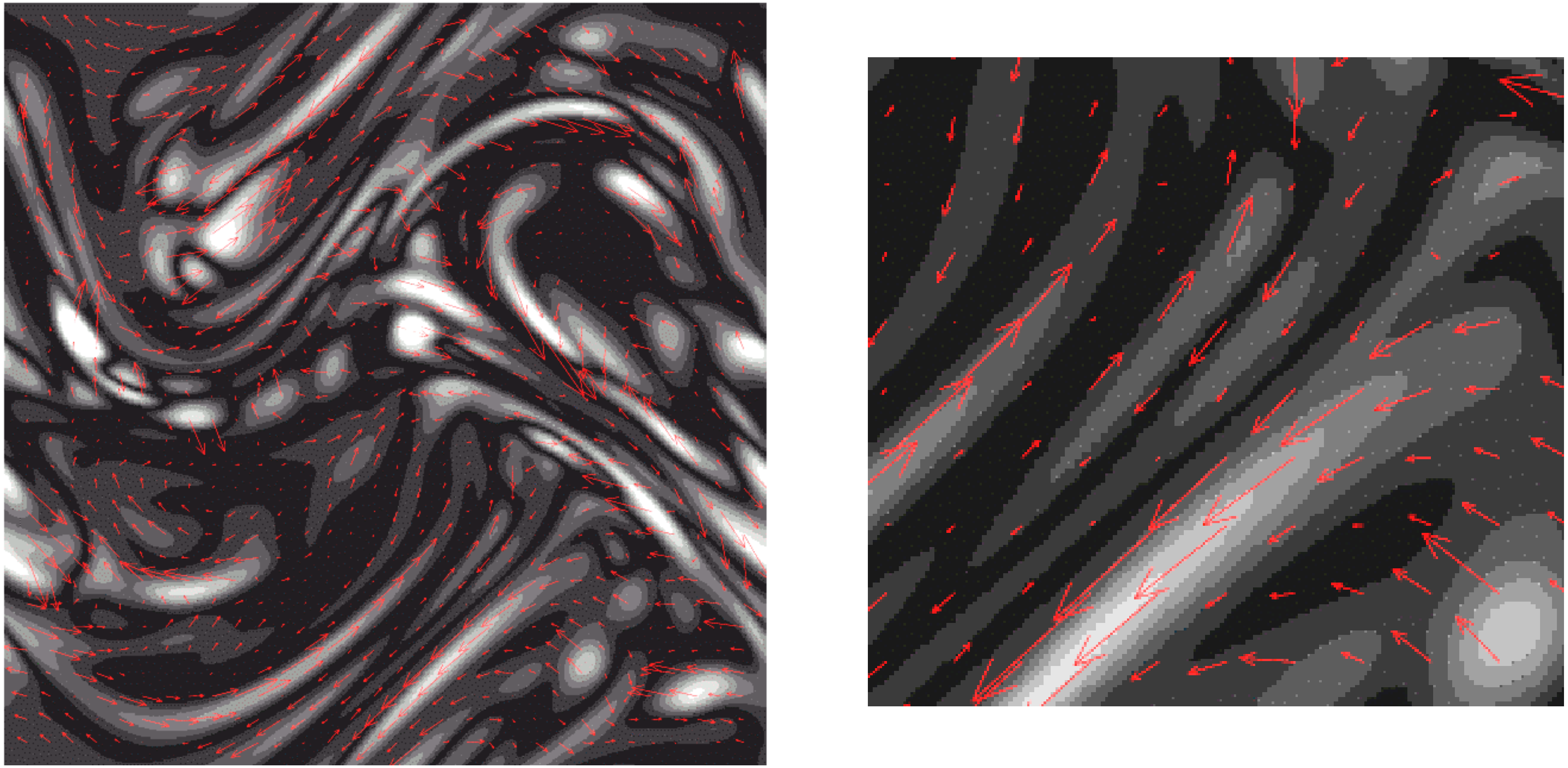
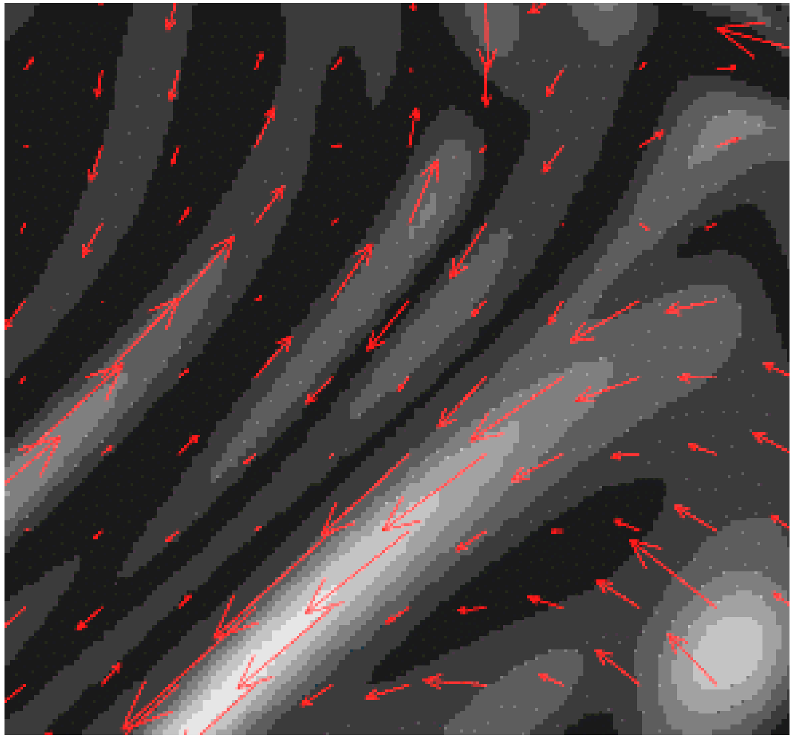


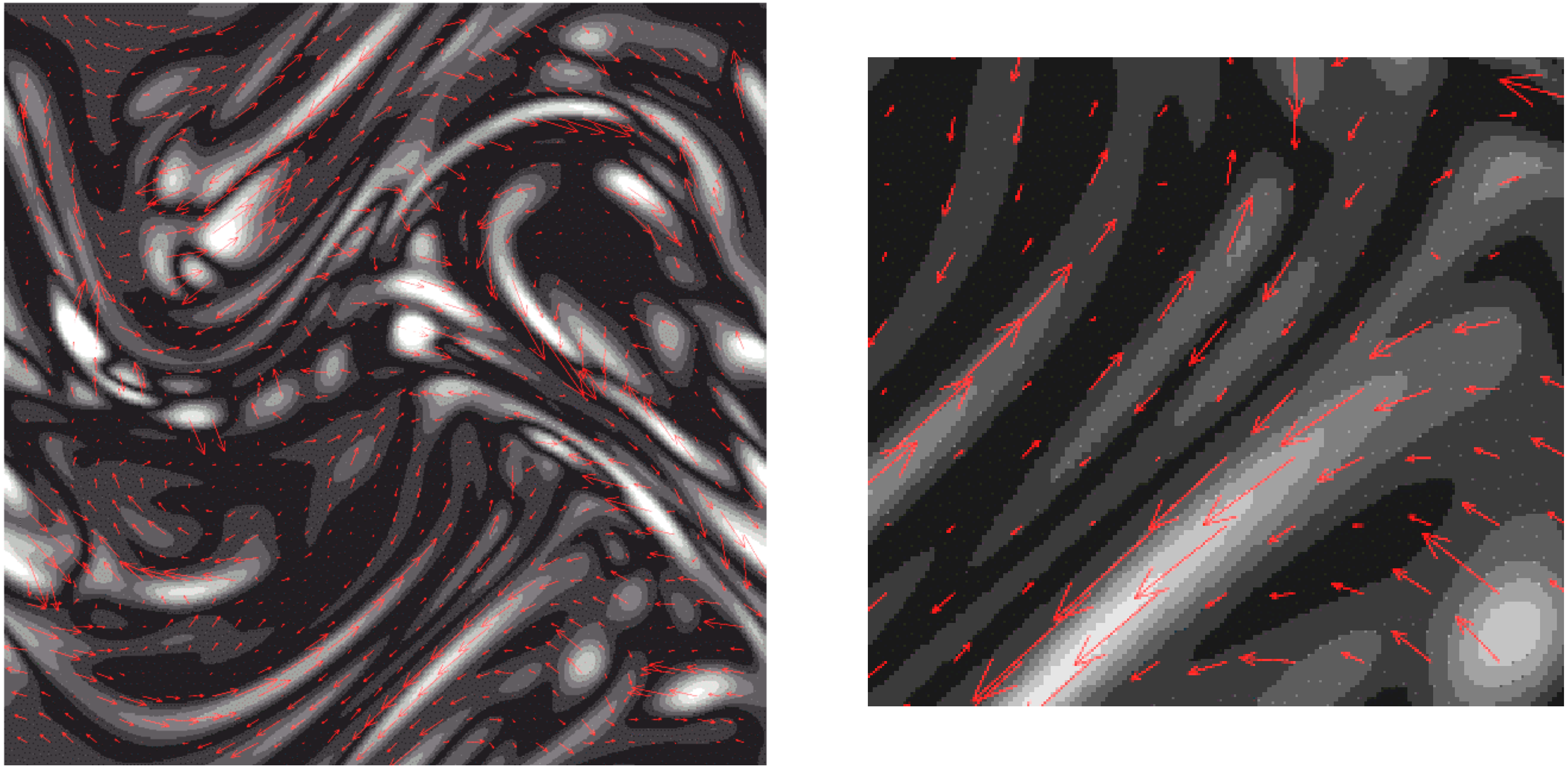
Folded Structure: The Movie

QuickTime™ and a
DV/DVCPRO - NTSC decompressor
are needed to see this picture.



In glorious black and white…

Diagnosing Folded Structure



Previous work:

- ***Extreme flux cancellation*** [Ott & coworkers 1988-98, Cattaneo 1994]
- ***Anisotropic two-point correlators*** [Chertkov *et al*. 1999, *PRL* **83**, 4065]

Characteristic Wavenumbers

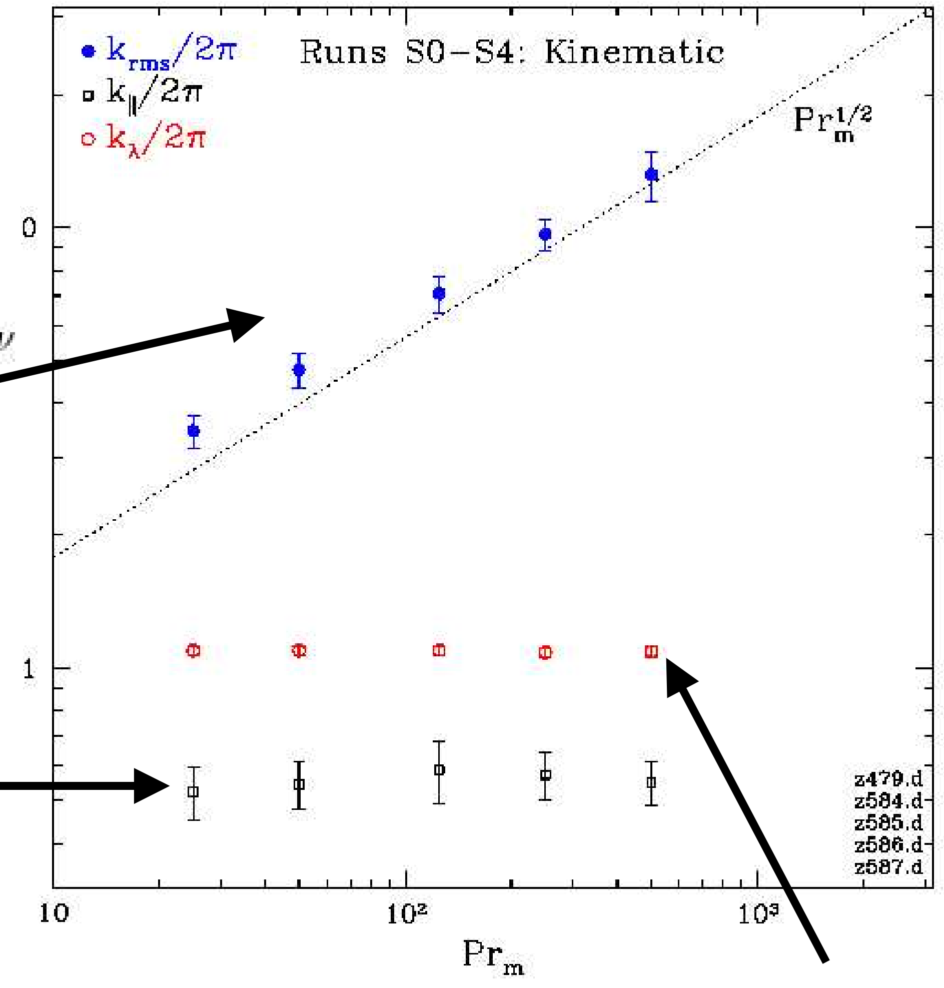
The crudest diagnostics:

$$k_{\rm rms} = \left[\frac{2}{\langle B^2\rangle}\int_0^\infty {\rm d}k\, k^2 M(k)\right]^{1/2} = \left[\frac{\langle|\nabla\mathbf{B}|^2\rangle}{\langle B^2\rangle}\right]^{1/2} \sim k_\eta \sim {\rm Pr}^{1/2} k_\nu$$

(**direction reversals**, get this already from spectral theory)

$$k_\parallel = \left[\frac{\langle|\mathbf{B}\cdot\nabla\mathbf{B}|^2\rangle}{\langle B^4\rangle}\right]^{1/2} \sim k_\nu$$

(inverse **“fold length”**, cannot get it from the spectrum!)



Compare them with the inverse Taylor microscale:

$$k_\lambda = \left[\frac{\langle|\nabla\mathbf{u}|^2\rangle}{\langle u^2\rangle}\right]^{1/2} = \frac{\sqrt{5}}{\lambda}$$

Folded Structure: Attempting Theory

There is a simple argument to show $k_{||} \sim k_{\nu}$:

consider **F**= **B**·∇**B** — it satisfies (throw away diffusion)

$$\partial_t \mathbf{F} + \mathbf{u}\cdot\nabla\mathbf{F} = \mathbf{F}\cdot\nabla\mathbf{u} + \mathbf{B}\mathbf{B} : \nabla\nabla\mathbf{u}$$

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Suppose $k_{||} >> k_{\nu}$ ⇒ neglect **∇∇u** ⇒ **F** satisfies same equation as **B**

Therefore $k_{||}^2 \sim \frac{\langle F^2 \rangle}{\langle B^4 \rangle} \propto \frac{\langle B^2 \rangle}{\langle B^4 \rangle} < \mathrm{const}\ \exp(-2\bar{\gamma}t)$ — decays!

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This argument makes no assumption about the flow,
but it is qualitative and resembles dangerously
the famous (incorrect) "Batchelor analogy,"
so we would like to do more quantitative theory.
We try Kazantsev model, of course.

Folded Structure from Kazantsev Model

$$\partial_t \mathrm{B} + \mathbf{u}\cdot\nabla\mathrm{B} = \mathrm{B}\cdot\nabla\mathbf{u}$$

$$\partial_t \mathbf{F} + \mathbf{u}\cdot\nabla\mathbf{F} = \mathbf{F}\cdot\nabla\mathbf{u} + \mathrm{BB} : \nabla\nabla\mathbf{u}$$

It is possible to derive ***an equation for the <u>joint PDF</u> of*** **B** ***and*** **F**:

$$\partial_t P = -\frac{1}{2}\kappa^{ij}_{,kl}\left(\frac{\partial}{\partial B^i}B^k + \frac{\partial}{\partial F^i}F^k\right)\left(\frac{\partial}{\partial B^j}B^l + \frac{\partial}{\partial F^j}F^l\right)P + \frac{1}{2}\kappa^{ij}_{,klmn}\frac{\partial^2}{\partial F^i \partial F^j}B^k B^l B^m B^n P$$

where (recall definitions)

$$\langle u^i(t,\mathbf{x})u^j(t',\mathbf{x}')\rangle = \delta(t-t')\kappa^{ij}(\mathbf{x}-\mathbf{x}')$$

$$\kappa^{ij}(\mathbf{y}) = \kappa_0\delta^{ij} - \frac{1}{2}\kappa_2\left(y^2\delta^{ij} - \frac{1}{2}y^i y^j\right) + \frac{1}{4!}\kappa_4 y^2\left(y^2\delta^{ij} - \frac{2}{3}y^i y^j\right) + \cdots$$

[AAS *et al.* 2002, *PRE* **65**, 016305]

Folded Structure from Kazantsev Model

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It is possible to derive an equation for the joint PDF of **B** and **F**.

This allows us to calculate all averages, individual and mixed:

thus, for example, $k_{\parallel}^2 \sim \frac{\langle F^2\rangle}{\langle B^4\rangle} \to \frac{14}{27}\frac{\kappa_4}{\kappa_2} \sim k_\nu^2$ **OK!**

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Now note that $\mathbf{F} = \mathbf{B}\cdot\nabla\mathbf{B} = B^2\left(\underbrace{\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}}}_{\substack{\mathbf{K}\\ \text{curvature}}} + \underset{\substack{\text{mirror}\\ \text{force}}}{\hat{\mathbf{b}}\frac{\nabla_\parallel B}{B}}\right)$

- Curvature **K** tells us about **the geometry of the field lines**
- Field strength *B* is **the density of the field lines**

Statistics of K and B give a description of field structure

[AAS *et al.* 2002, *PRE* **65**, 016305]

Curvature Statistics from Kazantsev Model

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{K} = \mathbf{K}\cdot(\nabla\mathbf{u})\cdot(\hat{\mathbb{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}}) - \hat{\mathbf{b}}\mathbf{K}\hat{\mathbf{b}} : \nabla\mathbf{u} - 2\mathbf{K}\hat{\mathbf{b}}\hat{\mathbf{b}} : \nabla\mathbf{u} + \hat{\mathbf{b}}\hat{\mathbf{b}} : (\nabla\nabla\mathbf{u})\cdot(\hat{\mathbb{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}})$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{b}} = \hat{\mathbf{b}}\cdot(\nabla\mathbf{u})\cdot\left(\hat{\mathbb{I}} - \hat{\mathbf{b}}\hat{\mathbf{b}}\right)$$

To get **curvature statistics**, again do a joint PDF and separate trivial parts (use isotropy):

$$P(t;\mathbf{K},\hat{\mathbf{b}}) = \delta(|\hat{\mathbf{b}}|^2 - 1)\delta(\hat{\mathbf{b}}\cdot\mathbf{K})\,\frac{1}{K}\,P_K(t;K)$$

[Drummond & Münch 1991, *JFM* **225**, 529; AAS *et al.* 2002, *PRE* **65**, 016305]

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Get $$\boxed{\partial_t P_K = \frac{7}{4}\kappa_2\frac{\partial}{\partial K}K\left[\left(1+K^2\right)\frac{\partial}{\partial K}\frac{1}{K}P_K + \frac{20}{7}P_K\right]}$$

K rescaled by $(2\kappa_4/7\kappa_2)^{1/2} \sim k_\nu$

[Drummond & Münch 1991, *JFM* **225**, 529; AAS *et al.* 2002, *PRE* **65**, 016305]

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There is a stationary solution:

$$P_K(K) = \frac{6}{7}\frac{K}{(1+K^2)^{10/7}} \quad \sim \text{power tail } K^{-13/7}$$

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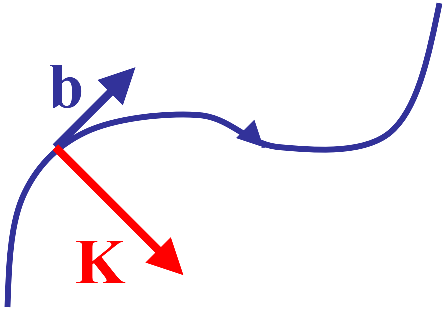
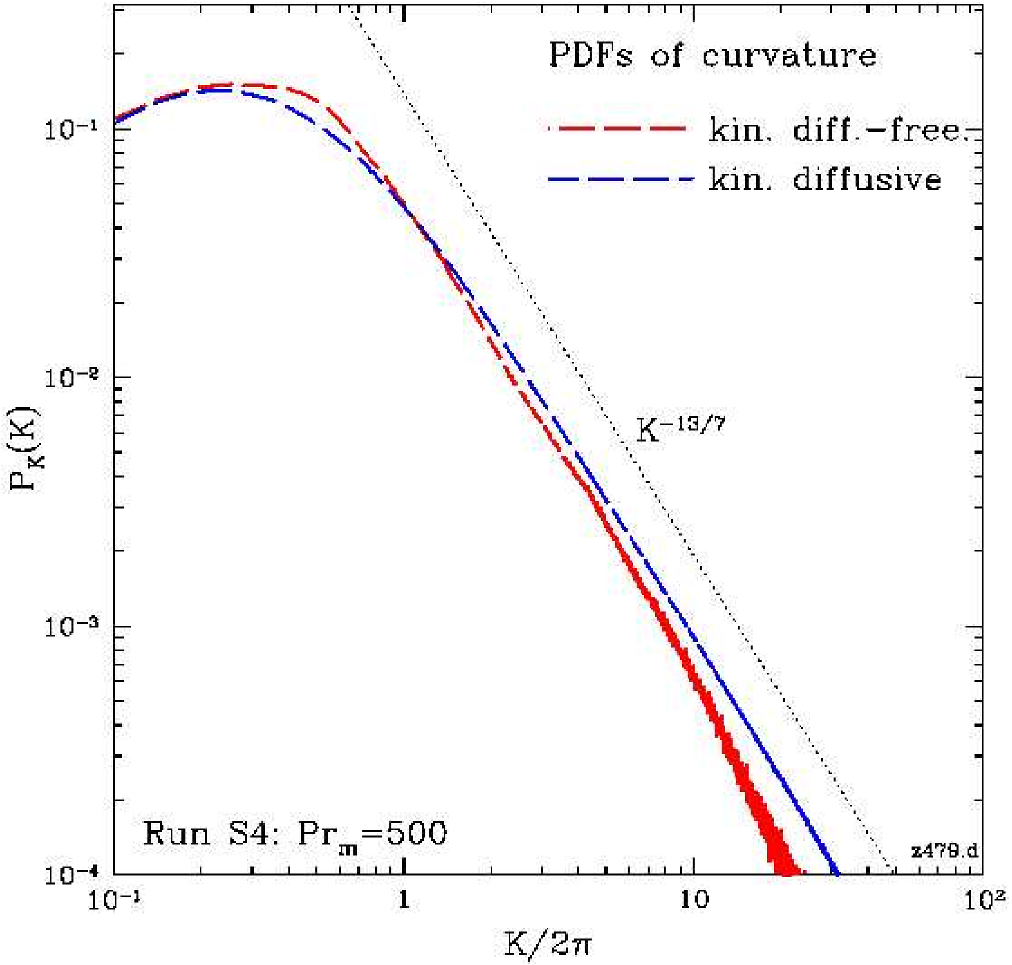
Field Line Curvature Statistics

More detailed information:
geometry of field lines described
by the PDF of their curvature
K=b·∇b





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cf. work on **material lines**: e.g., Drummond & Münch 1991, *JFM* **225**, 529]

Curvature and Field Strength

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geometry of field lines described
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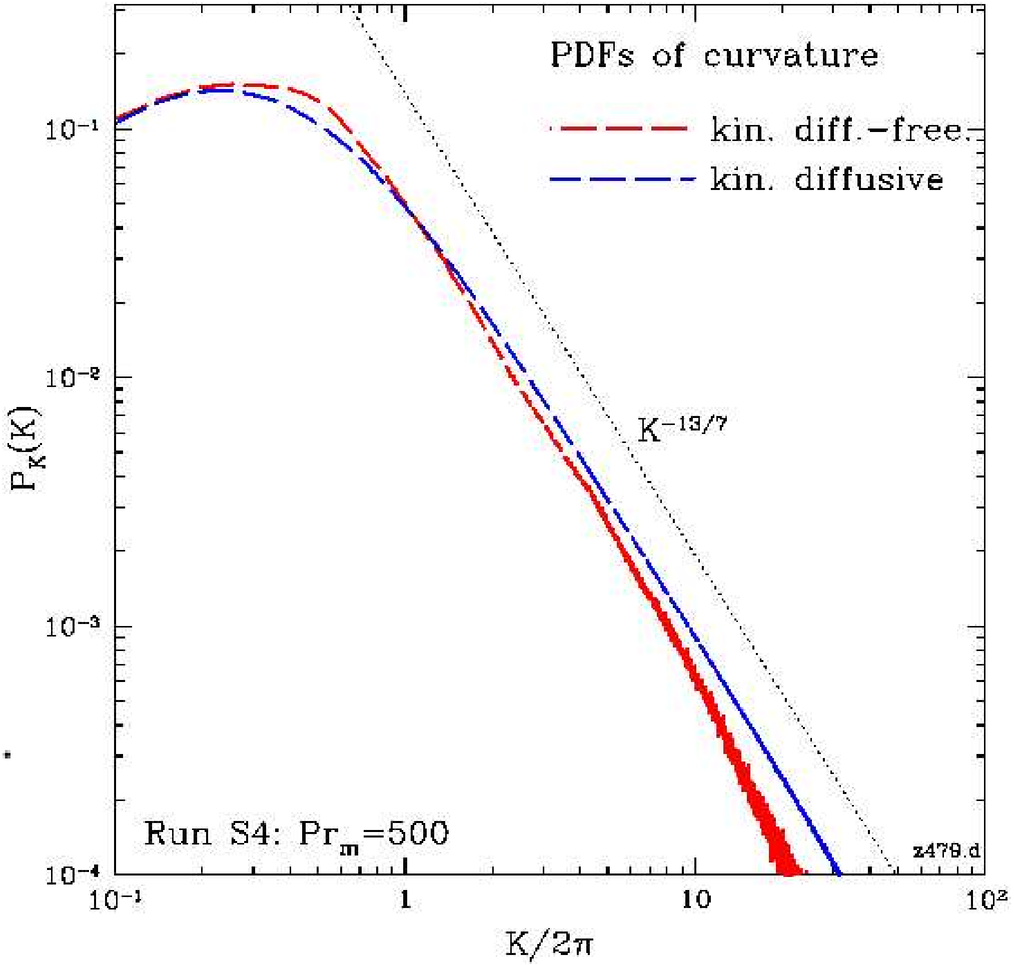
**Curvature and field strength
are anticorrelated**

$$\frac{\langle K^2B^2\rangle - \langle K^2\rangle\langle B^2\rangle}{\langle K^2\rangle\langle B^2\rangle} \simeq -0.9\ldots$$



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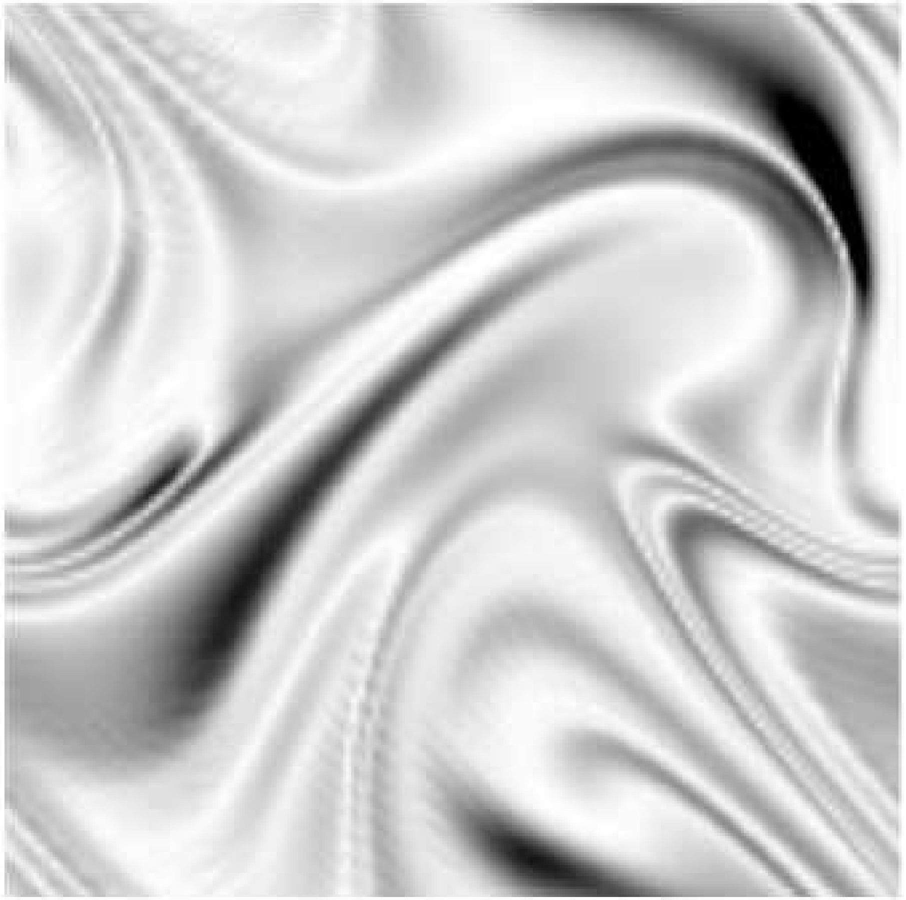

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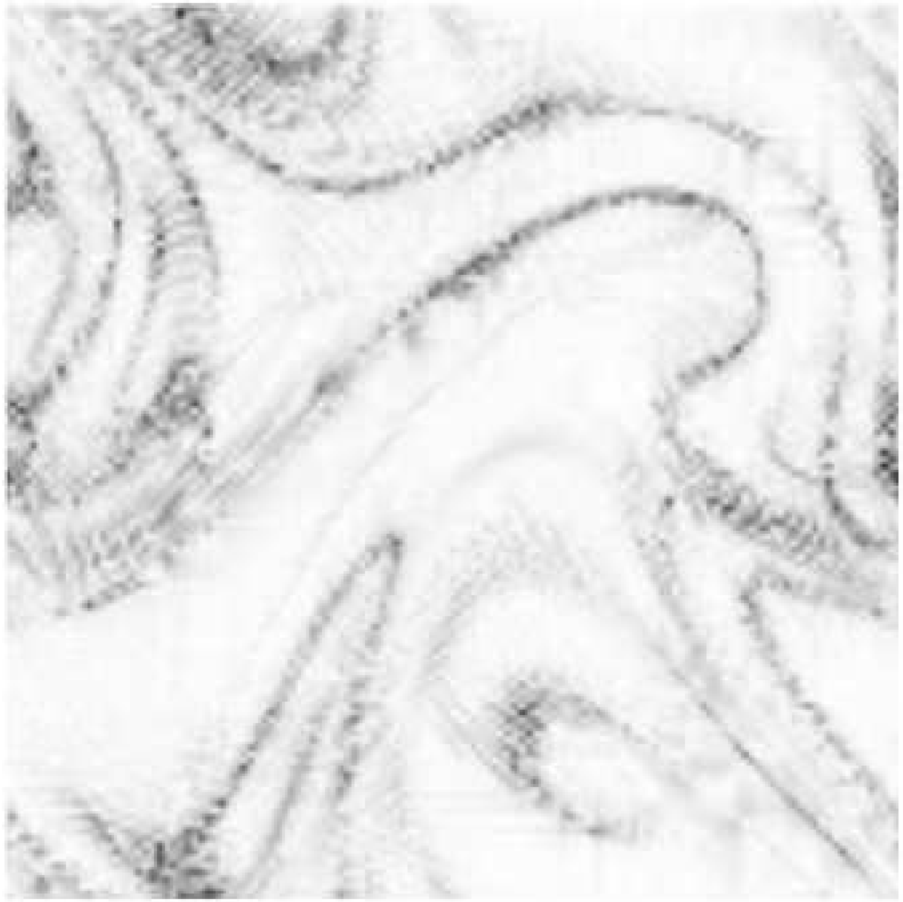

More detailed information: geometry of field lines described by the PDF of their curvature

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geometry of field lines described
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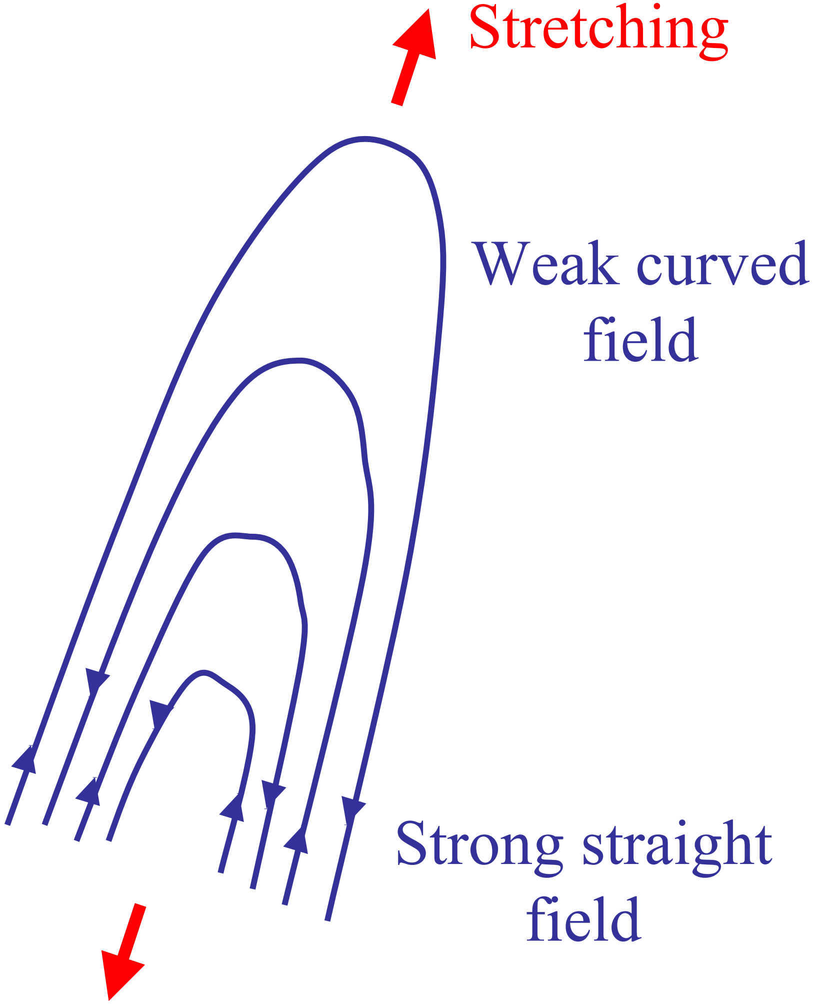
**Curvature and field strength
are anticorrelated**

…which is clear from simple
geometry of field stretching
(and can be shown both
analytically and
numerically)



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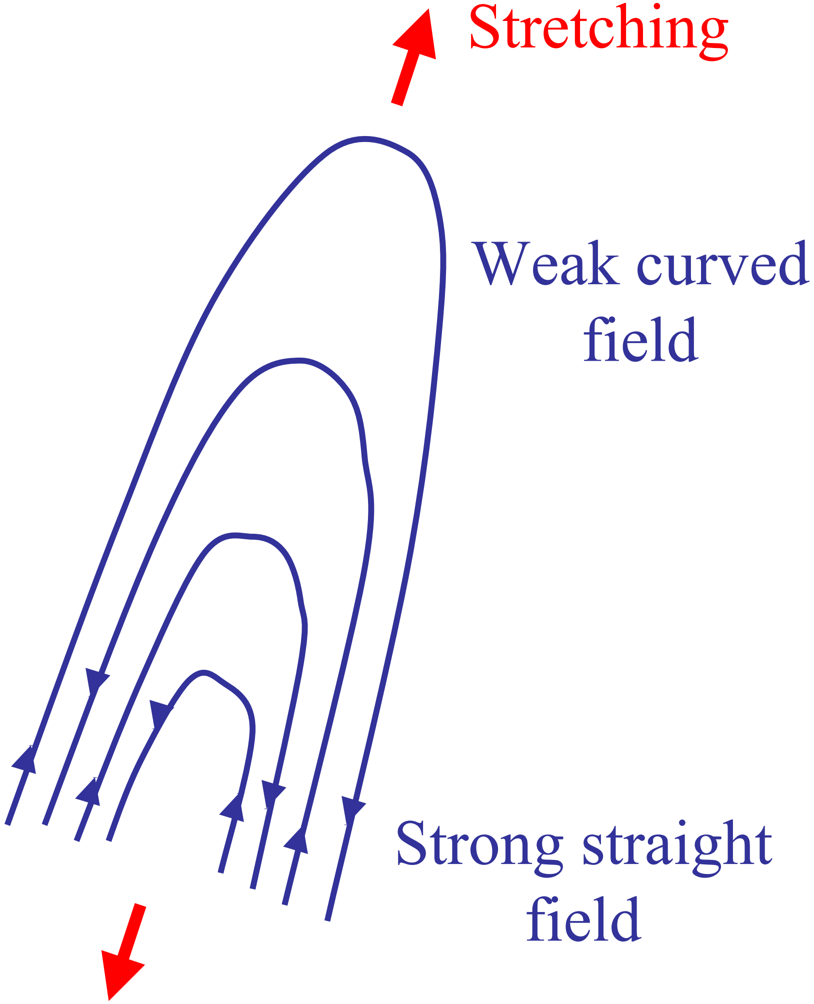
From Kazantsev model,
can show, for example,
that $\langle B^4K^2\rangle \sim k_\nu^2\langle B^4\rangle$,
but $\langle K^2\rangle \sim k_\nu^2\,\mathrm{Pr}^{4/7}$



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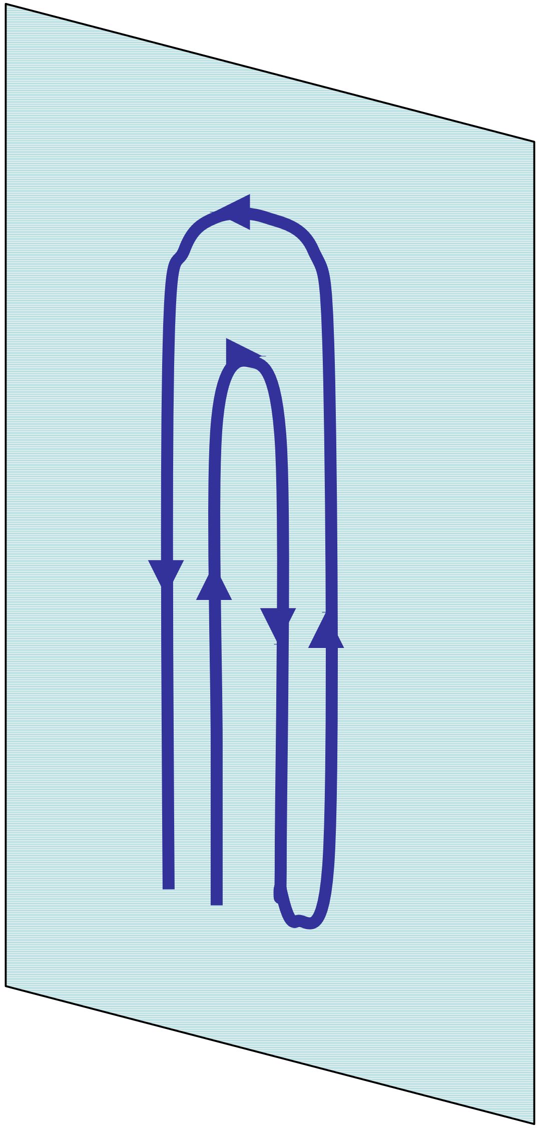
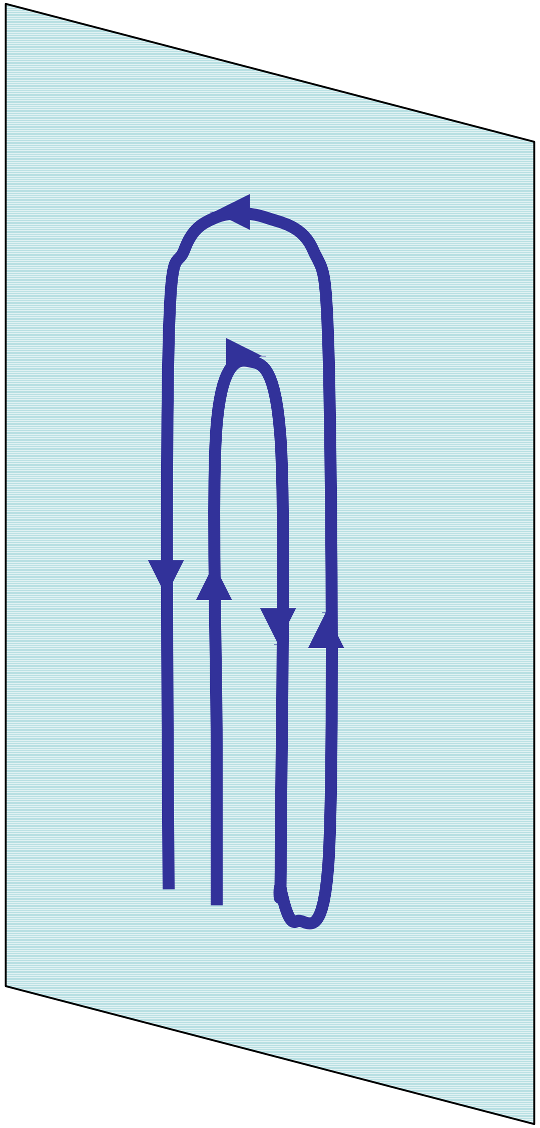
cf. work on **material lines**: e.g., Drummond & Münch 1991, *JFM* **225**, 529]

How Is Small-Scale Dynamo Possible?



How Is Small-Scale Dynamo Possible?

The Zeldovich *et al.* mechanism [*JFM* **144**, 1 (1984)]



Consider a <u>smooth</u> random flow
whose scale >> reversal scale

Locally, such a flow
can be thought of
as a ***random linear shear***,
with three Lyapunov directions:
stretching,
compression,
“null”

- viscous-scale eddies in turbulence
- deterministic chaotic flows
  (like ABC)
- Kazantsev model

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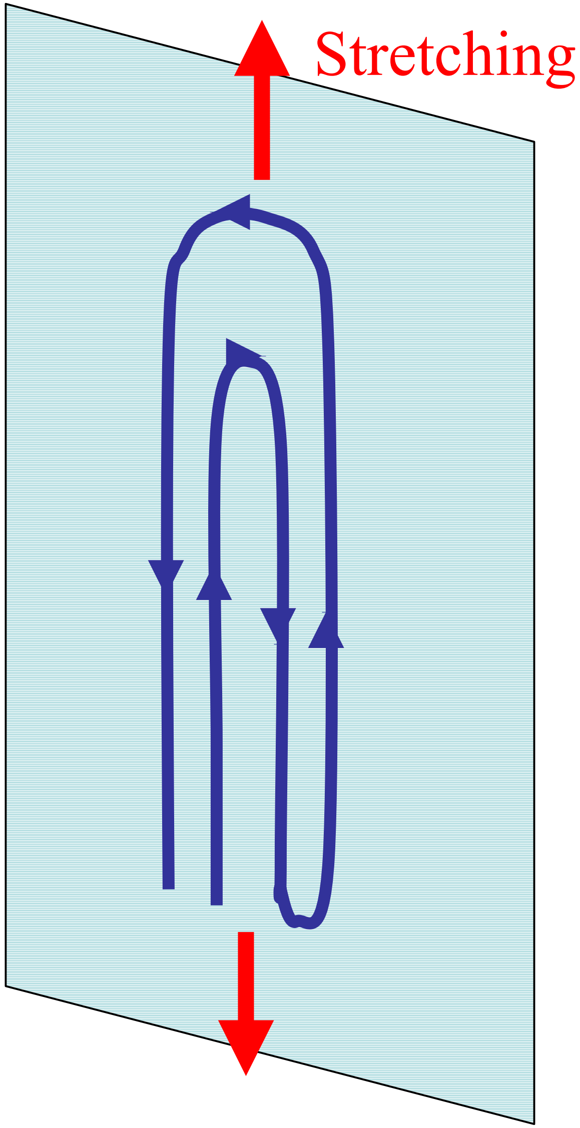

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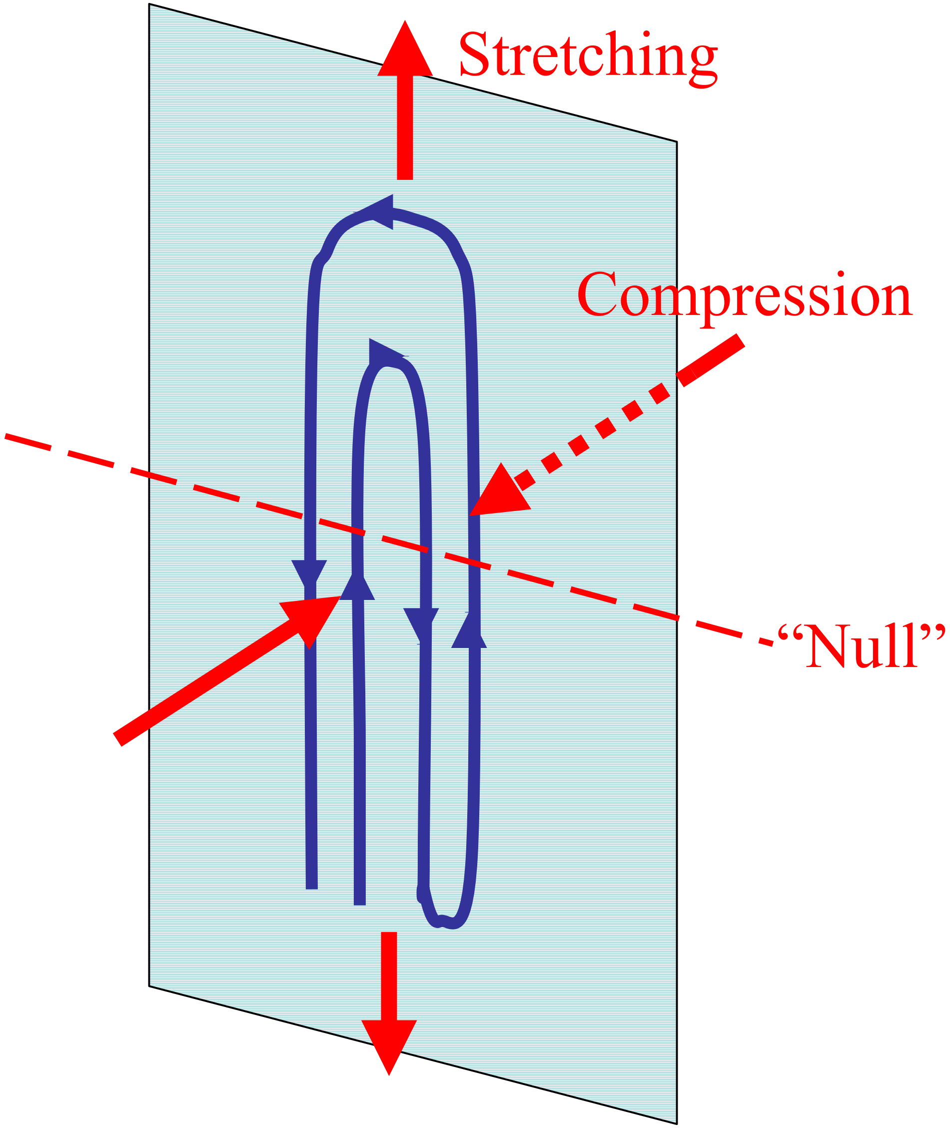

<u>Winning configuration</u>:
resistive annihilation of antiparallel fields does not cancel the effect of stretching

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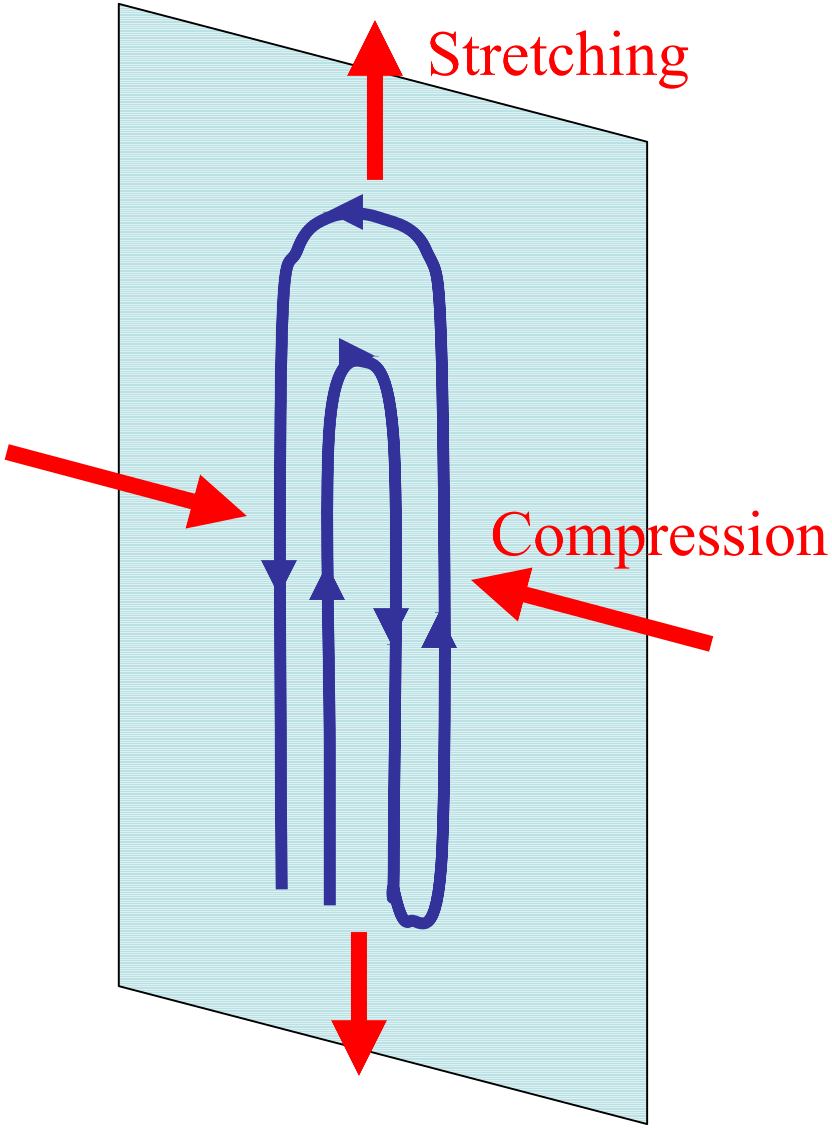
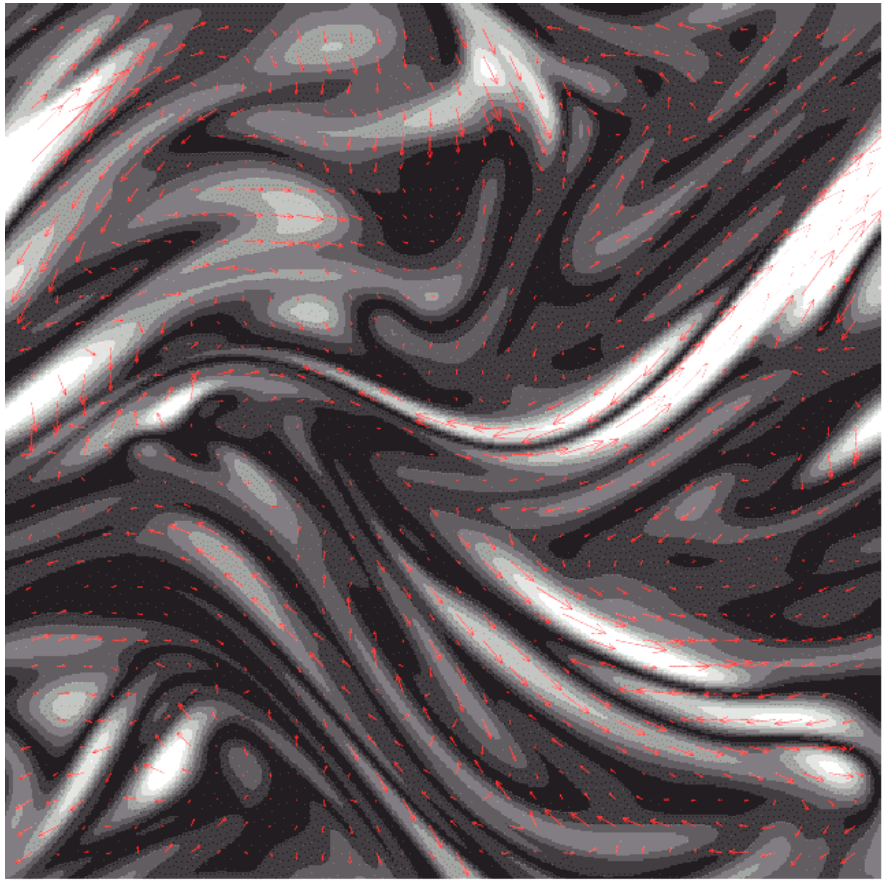
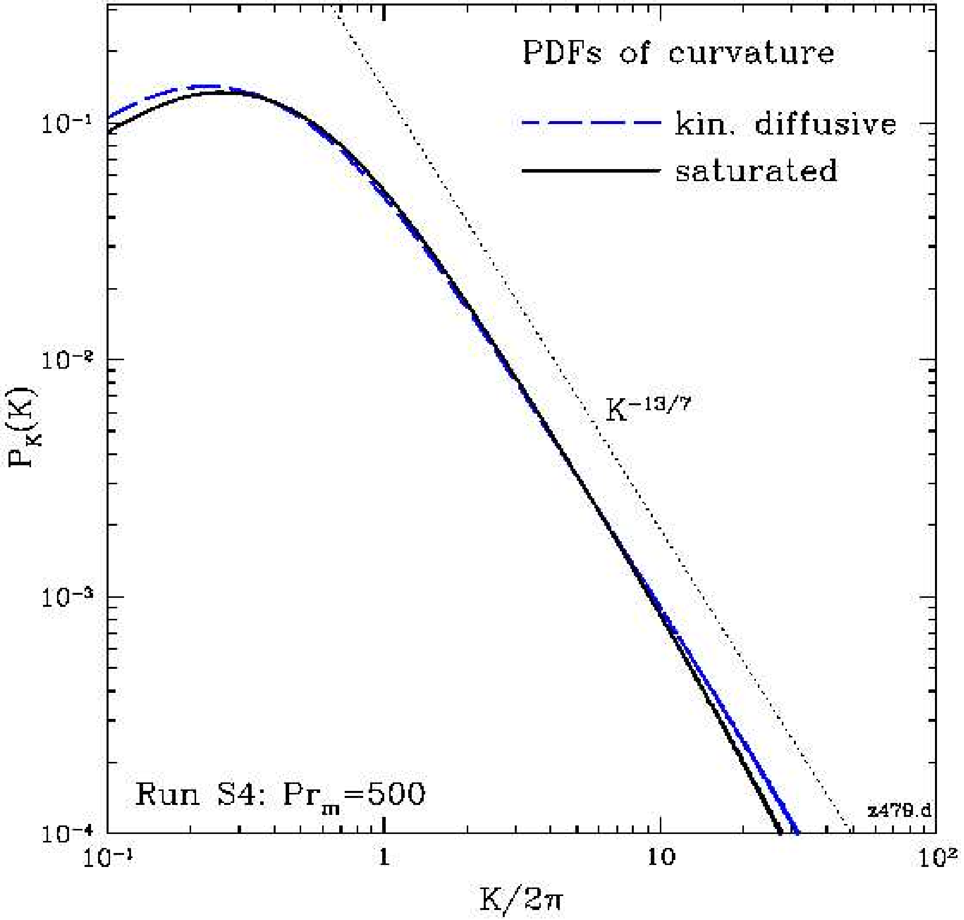


<u>Winning configuration</u>:
resistive annihilation of
antiparallel fields does
not cancel the effect of
stretching

This does not work in 2D!
[reason for antidynamo theorem,
Zeldovich 1956, *ZhETF* **31**, 154]

Folded Structure Preserved in Saturation

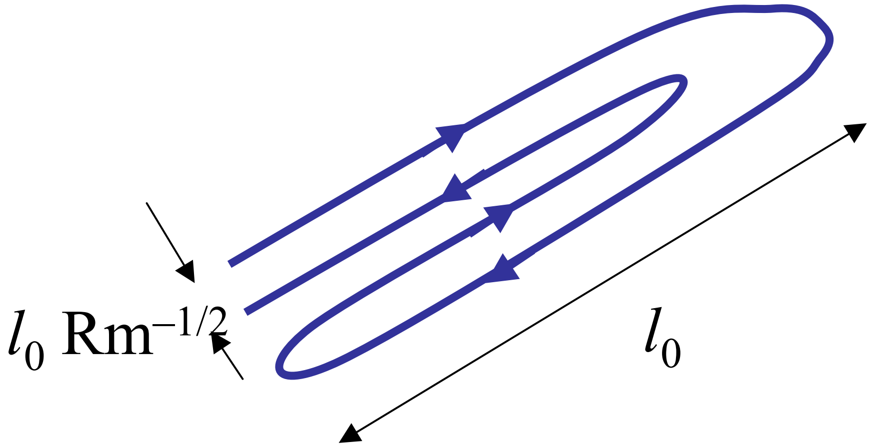




All the same features of **field-line geometry** and **field-strength anticorrelation with curvature** as in kinematic dynamo

[AAS *et al.* 2004, *ApJ* **612**, 276]

Saturation via Anisotropy



$$\mathbf{B}\cdot\nabla\mathbf{B} \sim k_{||}B^2$$

Folds provide a direction in space that is locally coherent at the scale of the flow

[AAS *et al.* 2004, *PRL* **92**, 084504]

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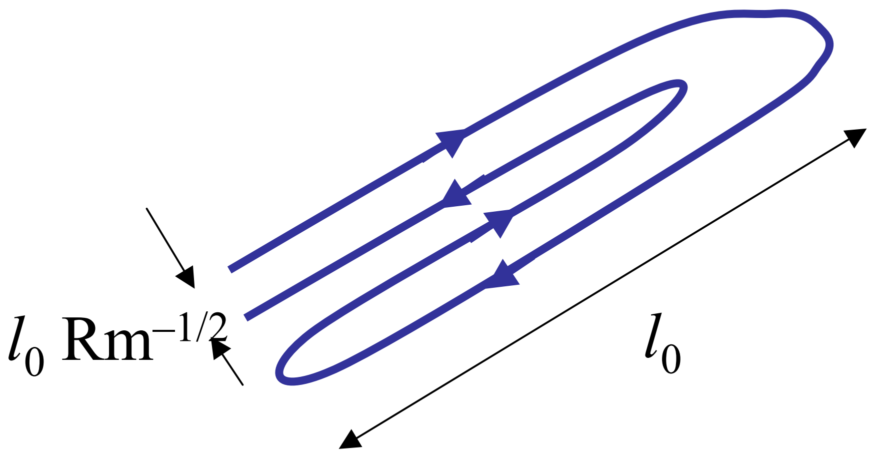


$\mathbf{B}\cdot\nabla\mathbf{B} \sim k_{||}B^2$

Folds provide a direction in space that is locally coherent at the scale of the flow

It is possible to construct a Fokker-Planck-type model of saturated spectra based on the idea that **saturation occurs via partial two-dimensionalisation of the velocity gradients with respect to the local direction of the folds**

This weakens stretching and enhances mixing, so dynamo saturates at marginally stable balance of the two

[AAS *et al.* 2004, *PRL* **92**, 084504]

A Fokker-Planck Model of Saturation

Build on the Kazantsev formalism and model saturation by

making velocity statistics depend on the local field direction $b^i b^j$:

$$\kappa^{ij}(\mathbf{k}) = \kappa^{(\mathrm{i})}(k,|\mu|)\left(\delta^{ij} - \hat{k}_i\hat{k}_j\right) + \kappa^{(\mathrm{a})}(k,|\mu|)\left(\hat{b}^i\hat{b}^j + \mu^2\hat{k}_i\hat{k}_j - \mu\hat{b}^i\hat{k}_j - \mu\hat{k}_i\hat{b}^j\right)$$

Can then derive an equation for magnetic-energy spectrum in (almost) the usual way:

$$\boxed{\partial_t M = \frac{1}{8}\gamma_\perp\frac{\partial}{\partial k}\left[(1+2\sigma_\|)k^2\frac{\partial}{\partial k} - (1+4\sigma_\perp+10\sigma_\|)\,k\right]M + 2(\sigma_\perp+\sigma_\|)\gamma_\perp M - 2\eta k^2 M}$$

where $\gamma_\perp = \int d^3k\,k_\perp^2\kappa_\perp \sim \left[\langle|\nabla_\perp\mathbf{u}_\perp|^2\rangle\right]^{1/2}$ “mixing rate”

$$\sigma_\perp = \frac{1}{\gamma_\perp}\int d^3k\,k_\|^2\kappa_\perp \sim \frac{\langle|\nabla_\|\mathbf{u}_\perp|^2\rangle}{\langle|\nabla_\perp\mathbf{u}_\perp|^2\rangle}, \quad \sigma_\| = \frac{1}{\gamma_\perp}\int d^3k\,k_\|^2\kappa_\| \sim \frac{\langle|\nabla_\|\mathbf{u}_\||^2\rangle}{\langle|\nabla_\perp\mathbf{u}_\perp|^2\rangle}$$

Solution in the limit $\eta \to +0$ is

$$\boxed{M(k) \simeq k^s e^{\gamma t} K_0(k/k_\eta),}$$

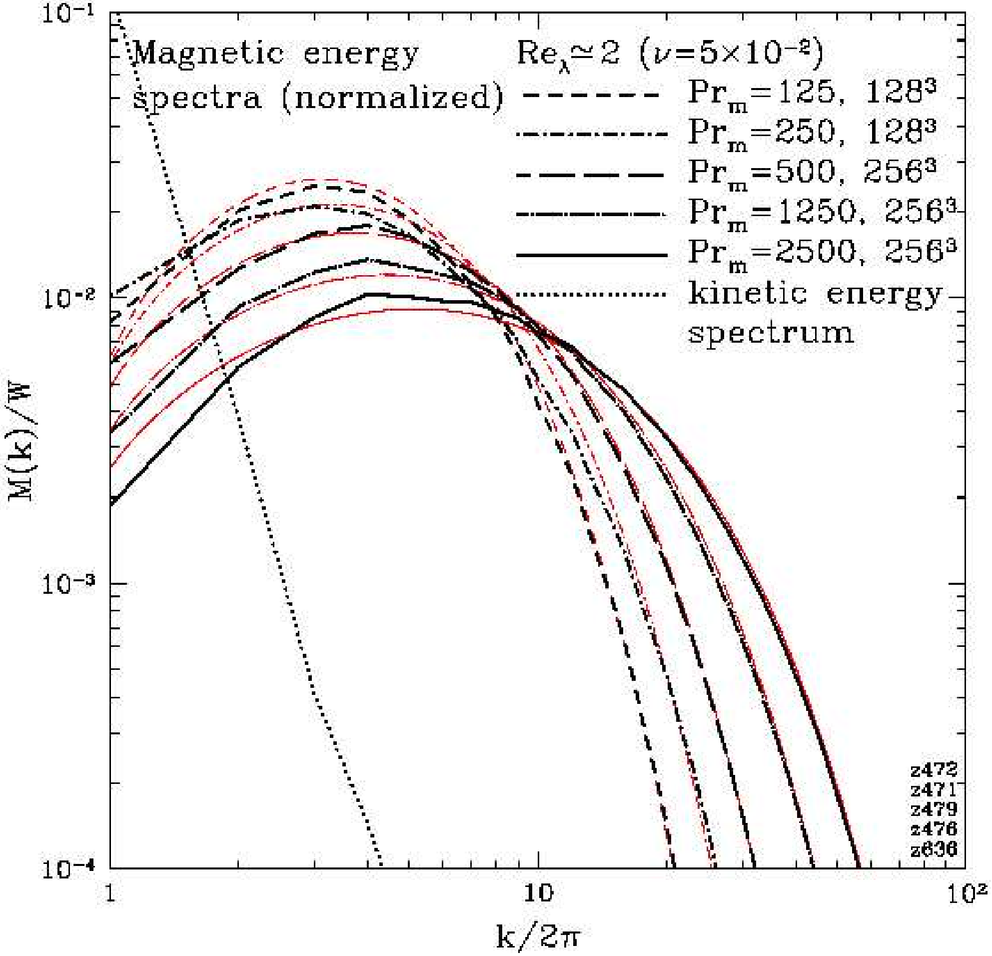
$$s = 2\frac{\sigma_\perp + 2\sigma_\|}{1+2\sigma_\|}, \quad \gamma = \gamma_\perp\left[2(\sigma_\perp+\sigma_\|) - \frac{(1+2\sigma_\perp+6\sigma_\|)^2}{8(1+2\sigma_\|)}\right], \quad k_\eta = \frac{1}{4}\left[\frac{(1+2\sigma_\|)\gamma_\perp}{\eta}\right]^{1/2}$$

γ= 0 at some sufficiently small $\sigma_\perp$, $\sigma_\|$ ⇒ saturation purely by means of anisotropy!

[AAS *et al.* 2004, *PRL* **92**, 084504]

Saturated Spectra: Theory vs. DNS

We can solve the model with simulation parameters: these nonasymptotic solutions fit an entire sequence of spectra in runs with Re ~ 1, Pm >>1



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Saturated Spectra: Theory vs. DNS

We can solve the model with simulation parameters:
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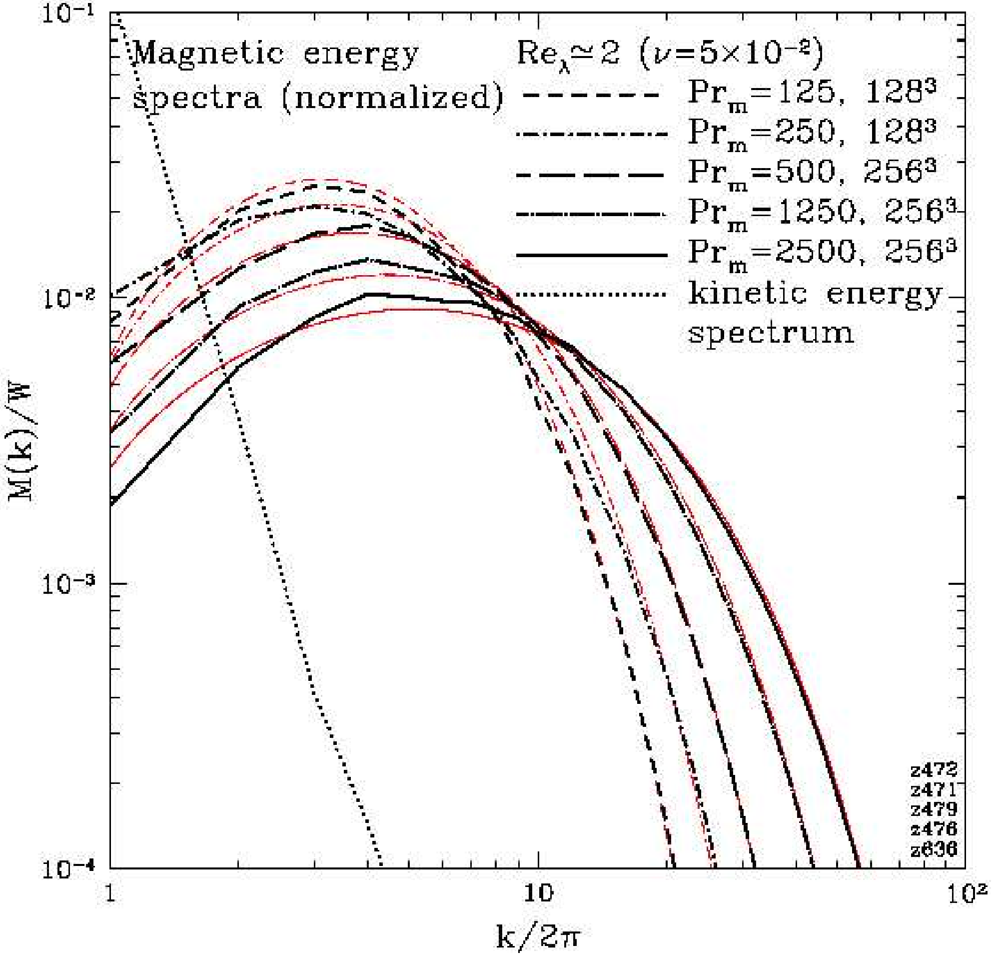
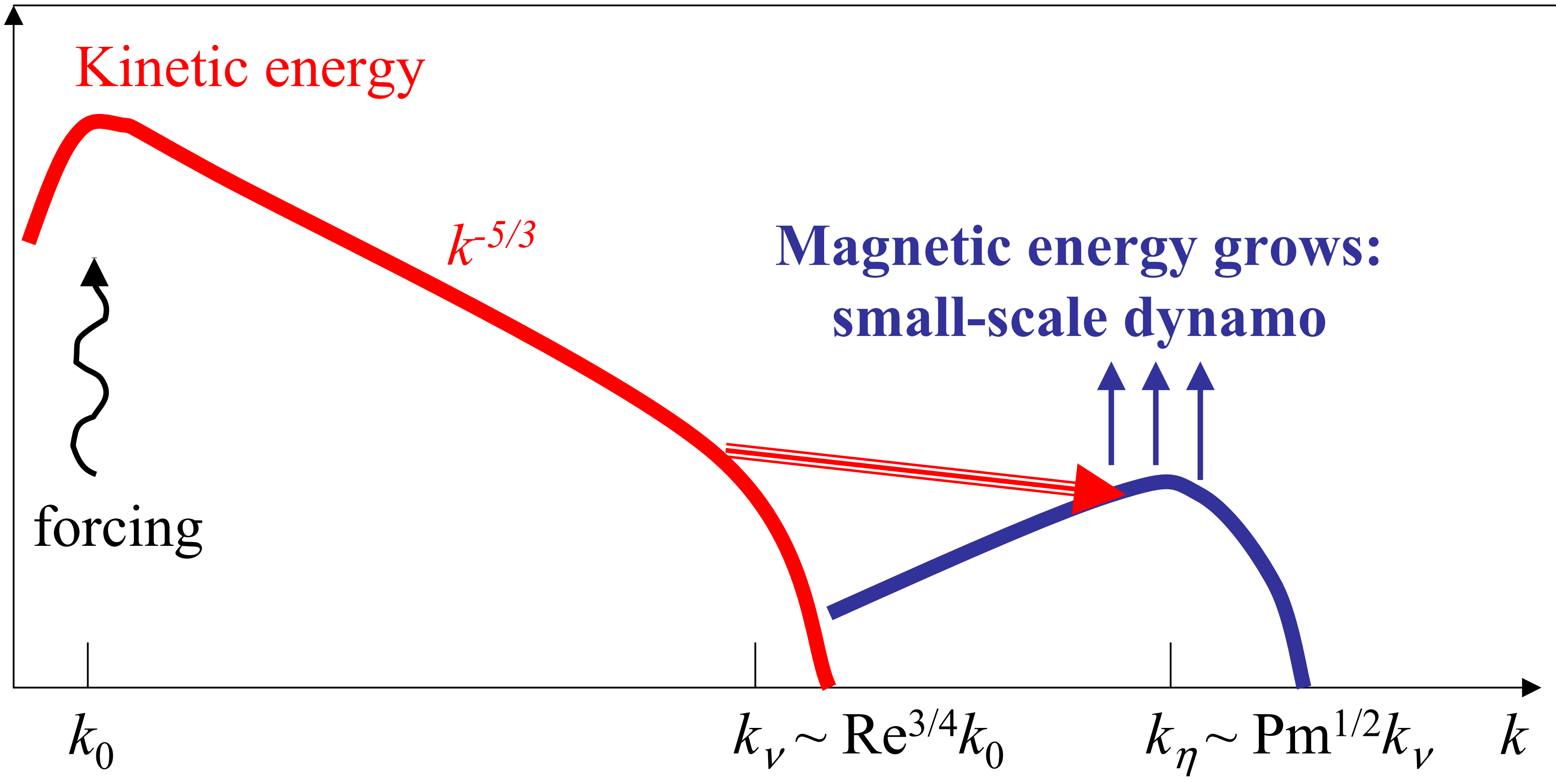
This is a pleasant surprise:
apparently, the saturation
mechanism is simple
and robust enough
to be captured by
such an elementary model!



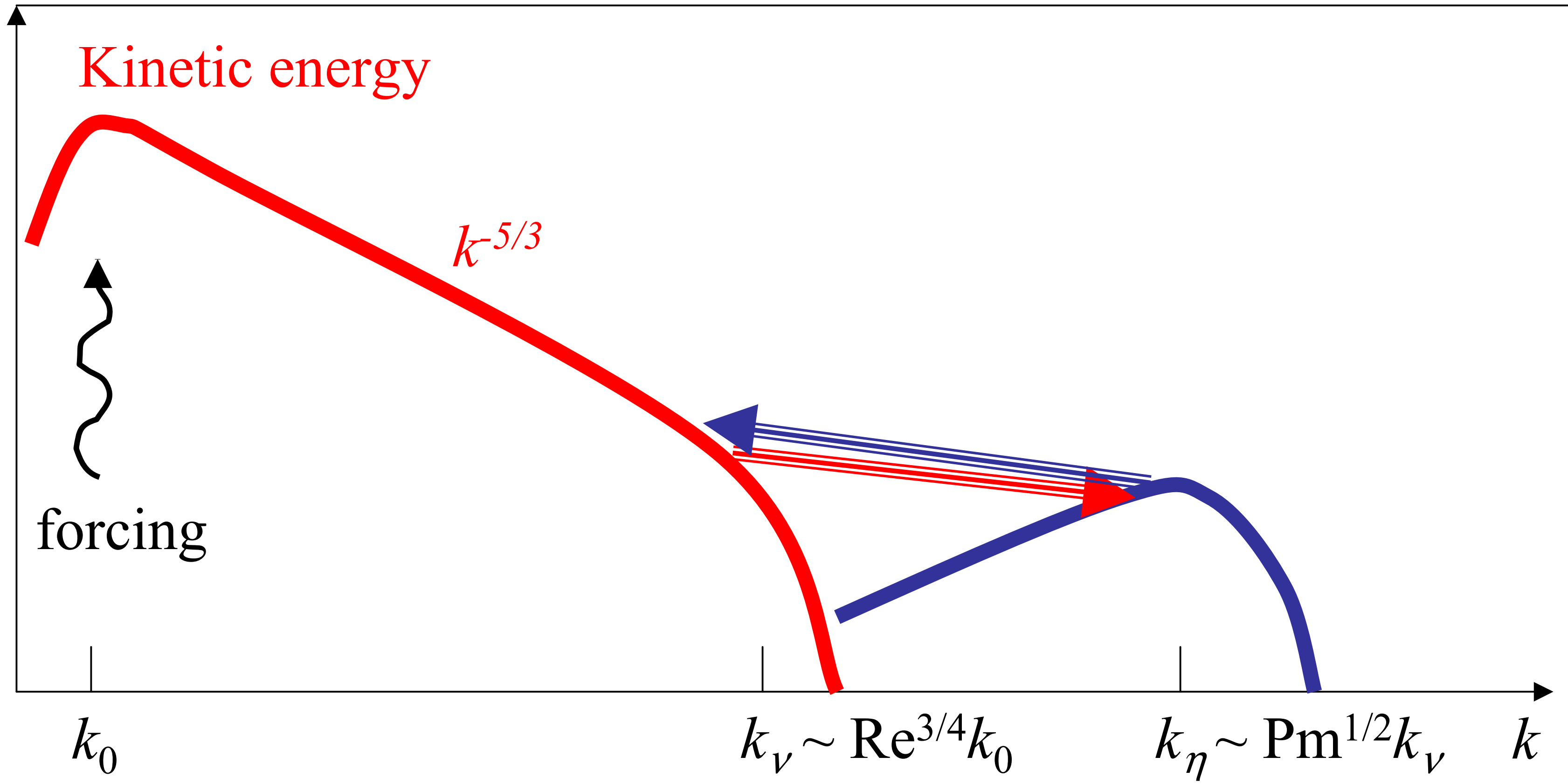
[AAS *et al.* 2004, *PRL* **92**, 084504]

MHD Turbulence: Multiscale Flow



We have thus far considered dynamo in a *single-scale* random flow
True turbulence has a range of scales

Onset of Back Reaction



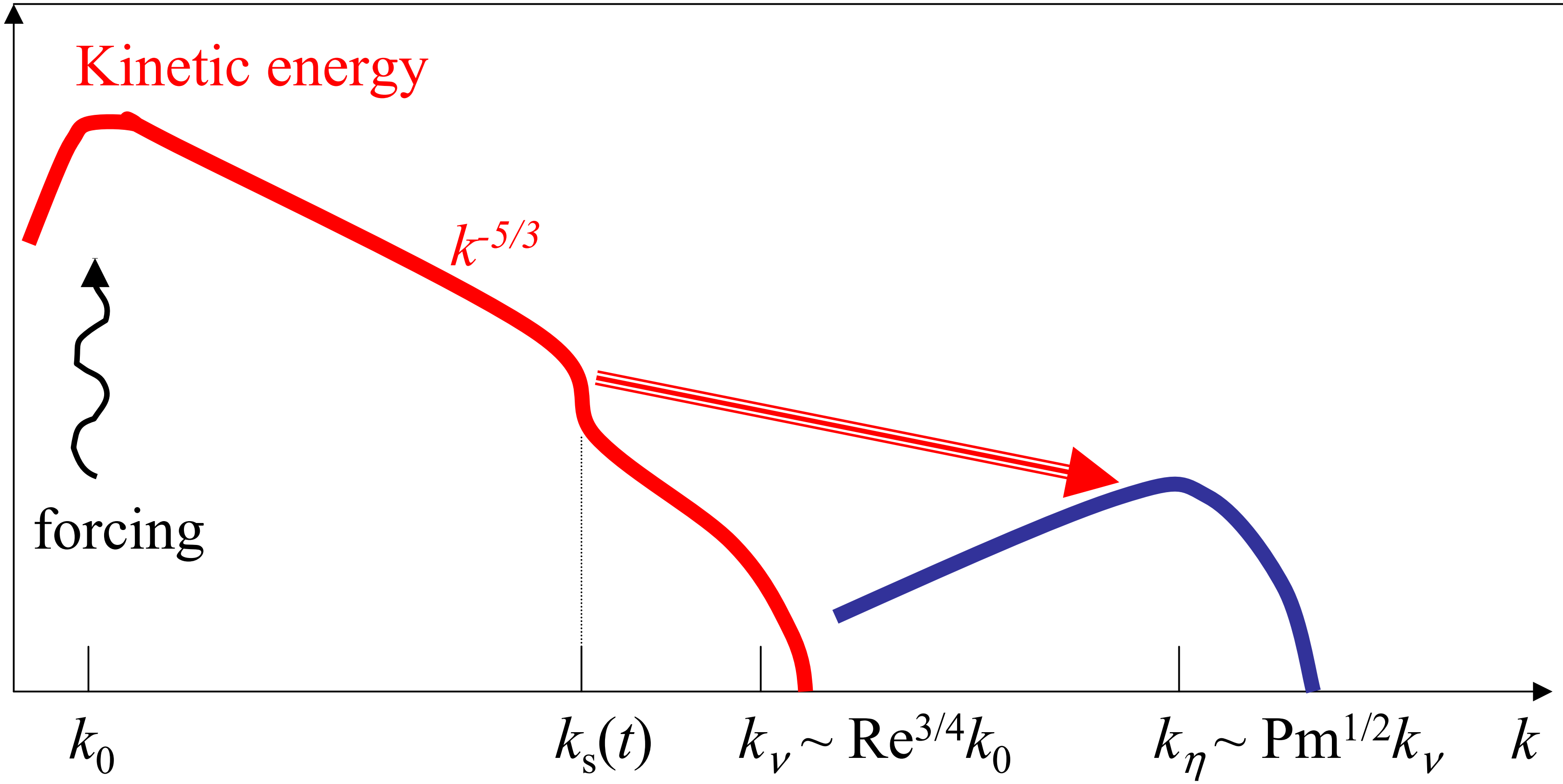
Kinematic growth
continues until $\mathbf{B}\cdot\nabla\mathbf{B} \sim \mathbf{u}\cdot\nabla\mathbf{u}$

$k_{||} B^2 \sim k_\nu u^2$ i.e., $B^2 \sim u^2$

Mag. energy ~ visc. eddies energy

[AAS *et al.* 2002, *PRE* **65**, 016305]

Intermediate Nonlinear Growth



Define *stretching scale* $l_\mathrm{s}(t)$: $u^2_{\ell_s} \sim \langle B^2 \rangle$

[AAS *et al*. 2002, *NJP* **4**, 84; Maron *et al.* 2004, *ApJ* **603**, 569]

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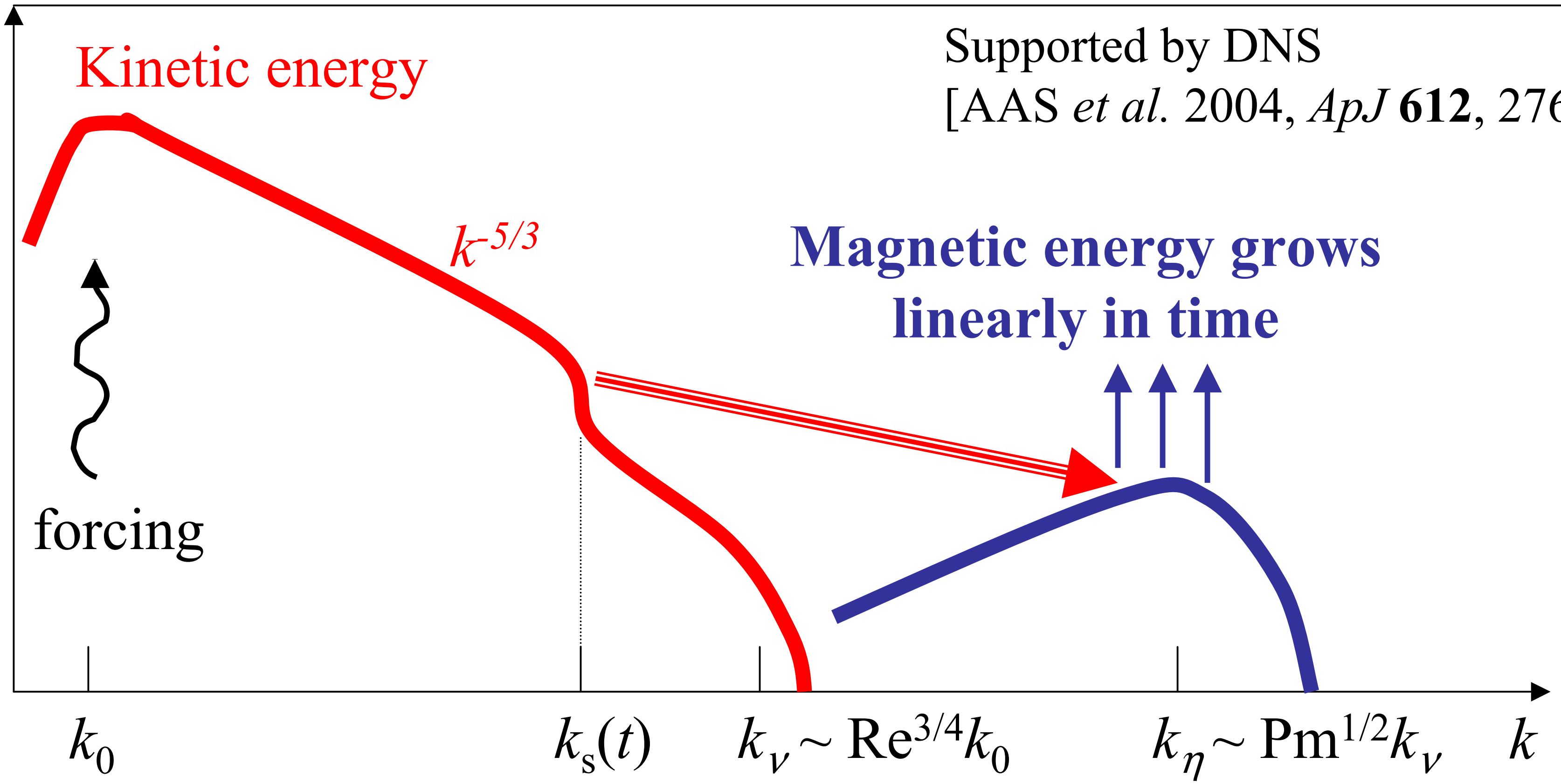

Supported by DNS
[AAS *et al.* 2004, *ApJ* **612**, 276]

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$$\frac{d}{dt}\langle B^2\rangle \sim \frac{u_{\ell_s}}{\ell_s}\langle B^2\rangle \sim \frac{u_{\ell_s}^3}{\ell_s} \sim \epsilon = \mathrm{const} \longrightarrow \boxed{\langle B^2\rangle(t) \sim \epsilon t}$$

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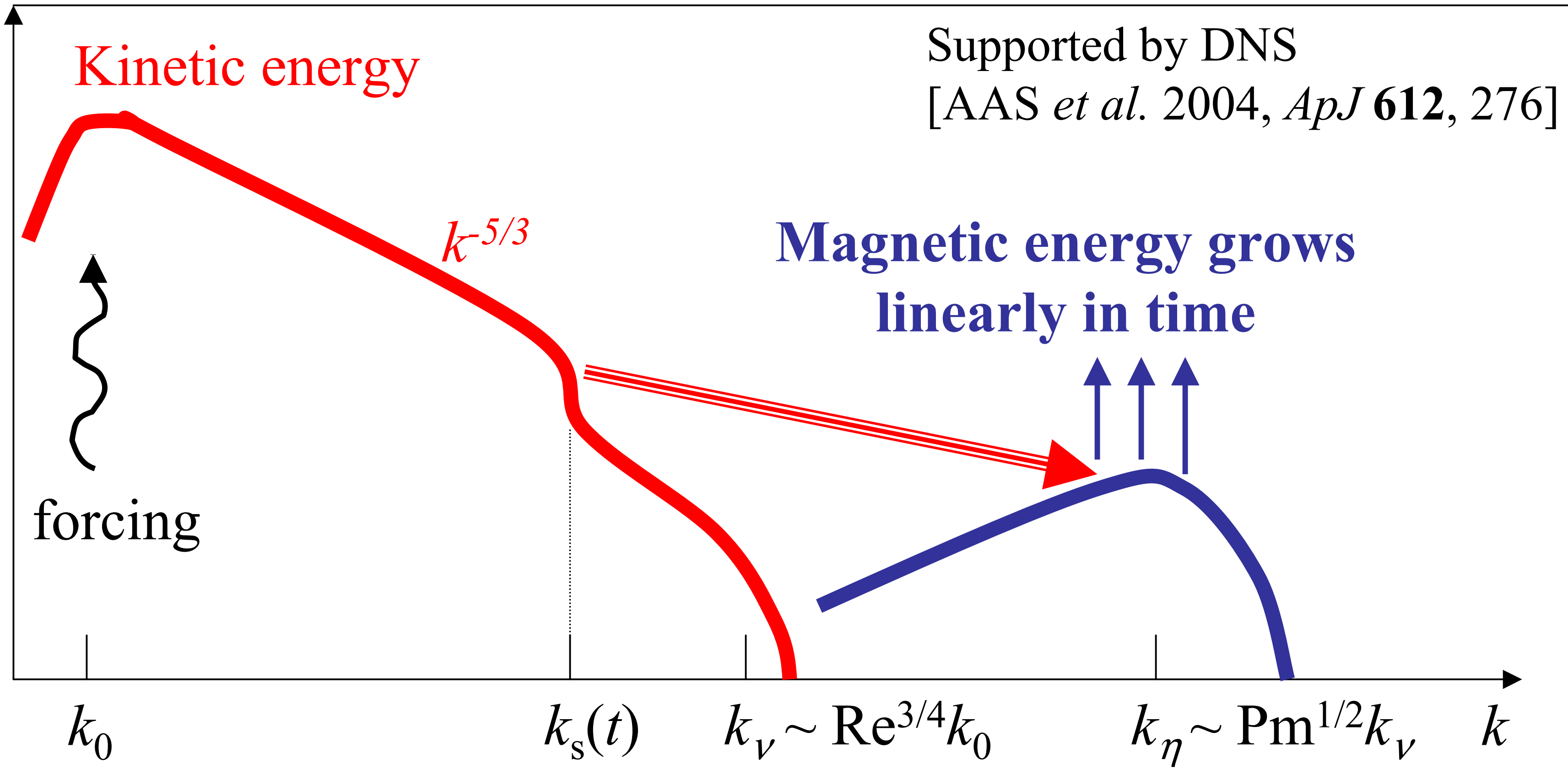

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Magnetic energy grows linearly in time

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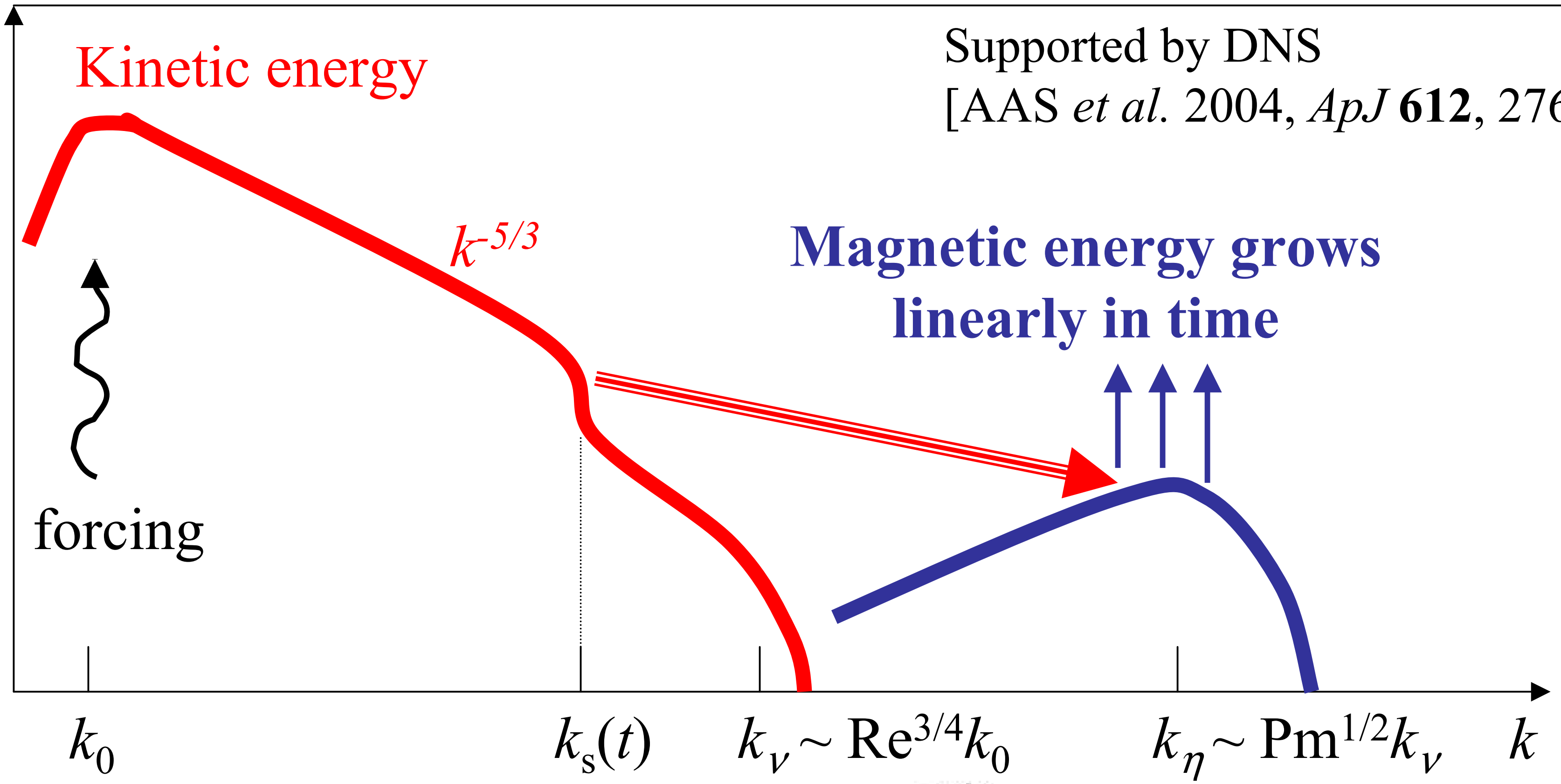
$$\mathbf{u}\cdot\nabla\mathbf{u} \sim \frac{u_{\ell_s}^2}{\ell_s} \sim \mathbf{B}\cdot\nabla\mathbf{B} \sim k_{\|}\langle B^2\rangle \longrightarrow k_{\|} \sim \ell_s^{-1} \sim \epsilon^{-1/2} t^{-3/2}$$

Intermediate Nonlinear Growth

Supported by DNS
[AAS *et al*. 2004, *ApJ* **612**, 276]



Kinetic energy

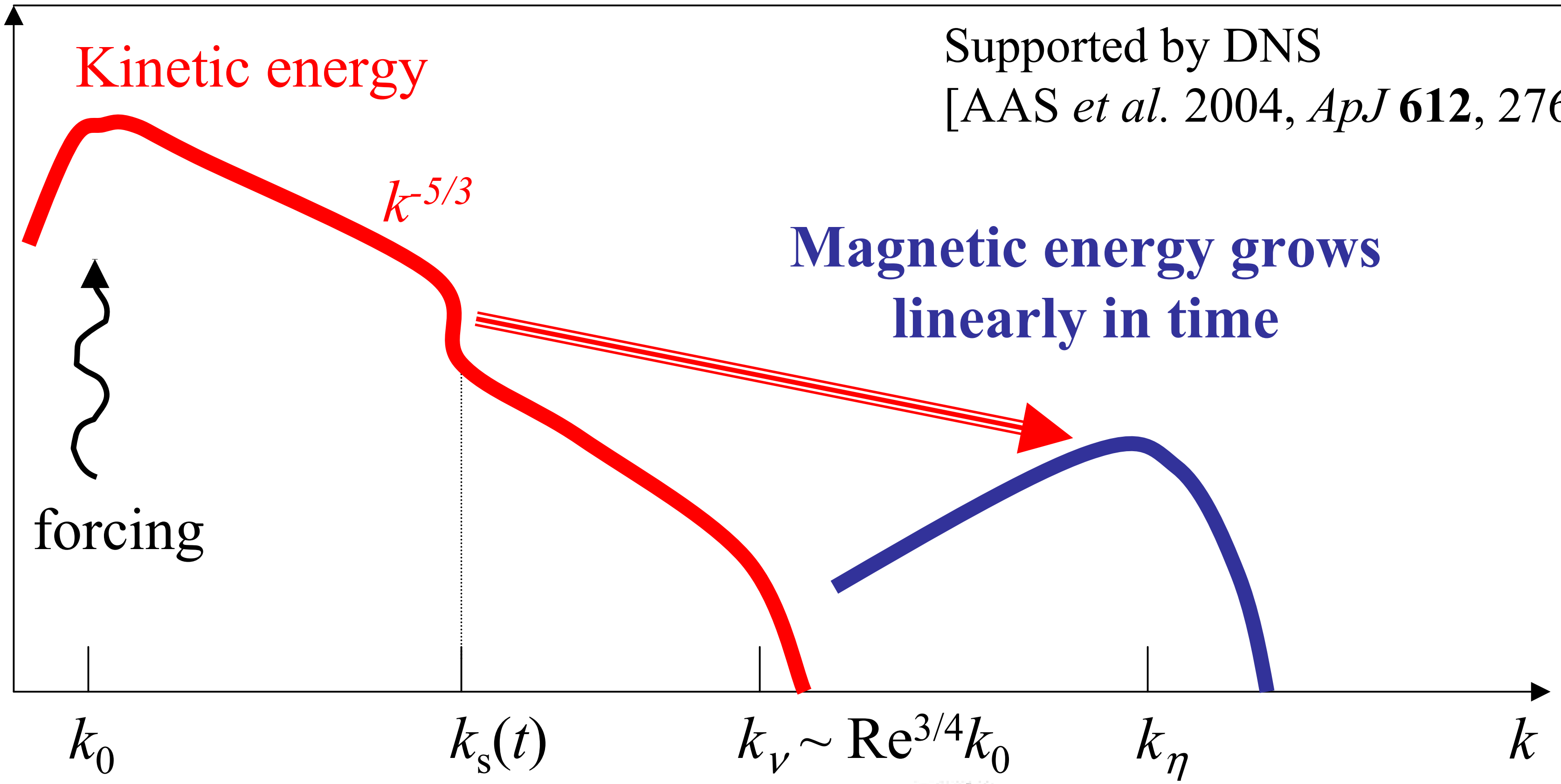
Magnetic energy grows linearly in time

$$\frac{u_{\ell_s}}{\ell_s} \sim \frac{\eta}{\ell_\eta^2} \longrightarrow \ell_\eta \sim \left(\frac{\eta}{\ell_s/u_{\ell_s}}\right)^{1/2} \sim (\eta t)^{1/2}$$ **selective decay**

It is possible to construct a Fokker-Planck model of this self-similar intermediate growth stage

[AAS *et al*. 2002, *NJP* **4**, 84]

Intermediate Nonlinear Growth



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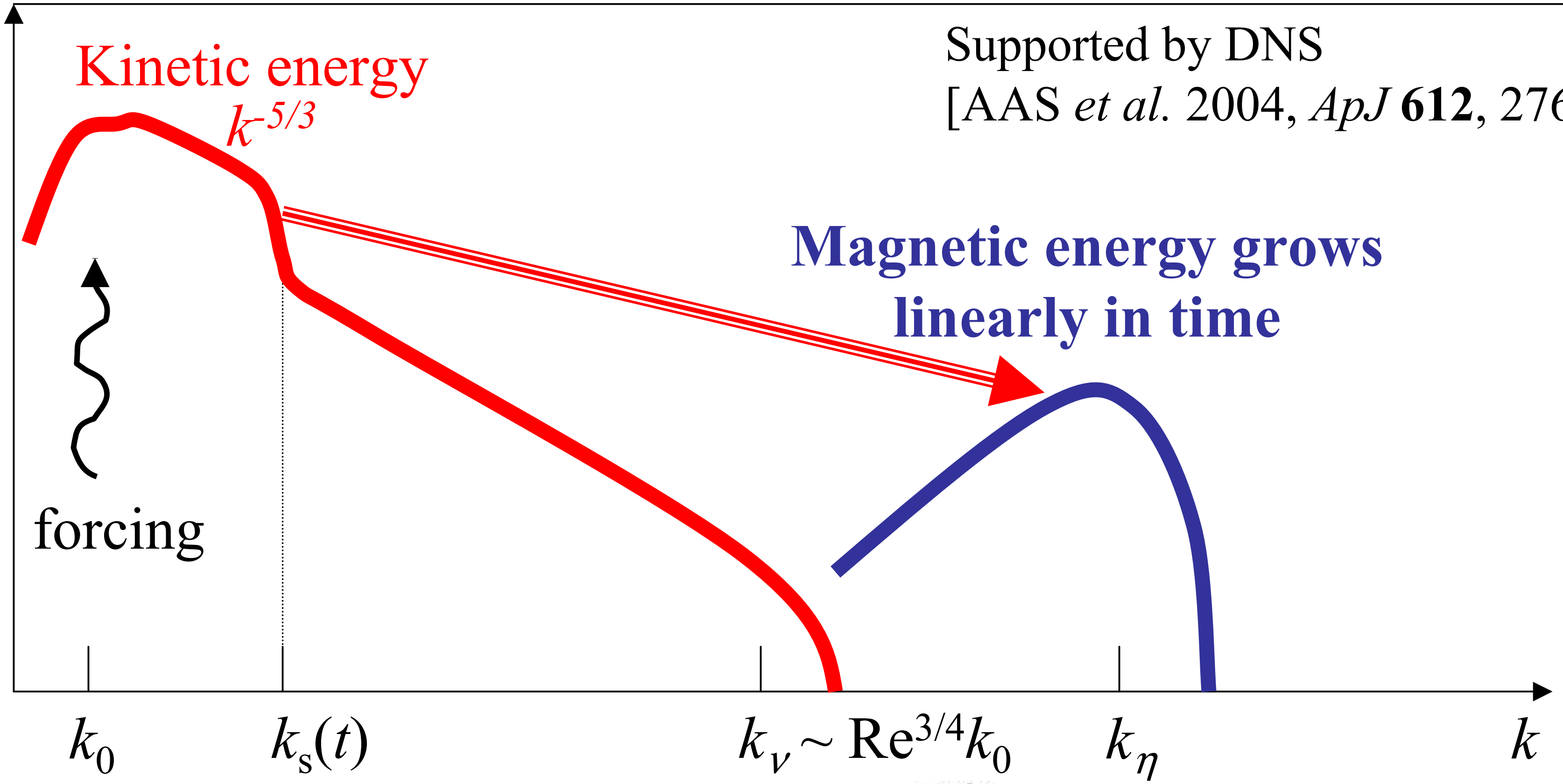

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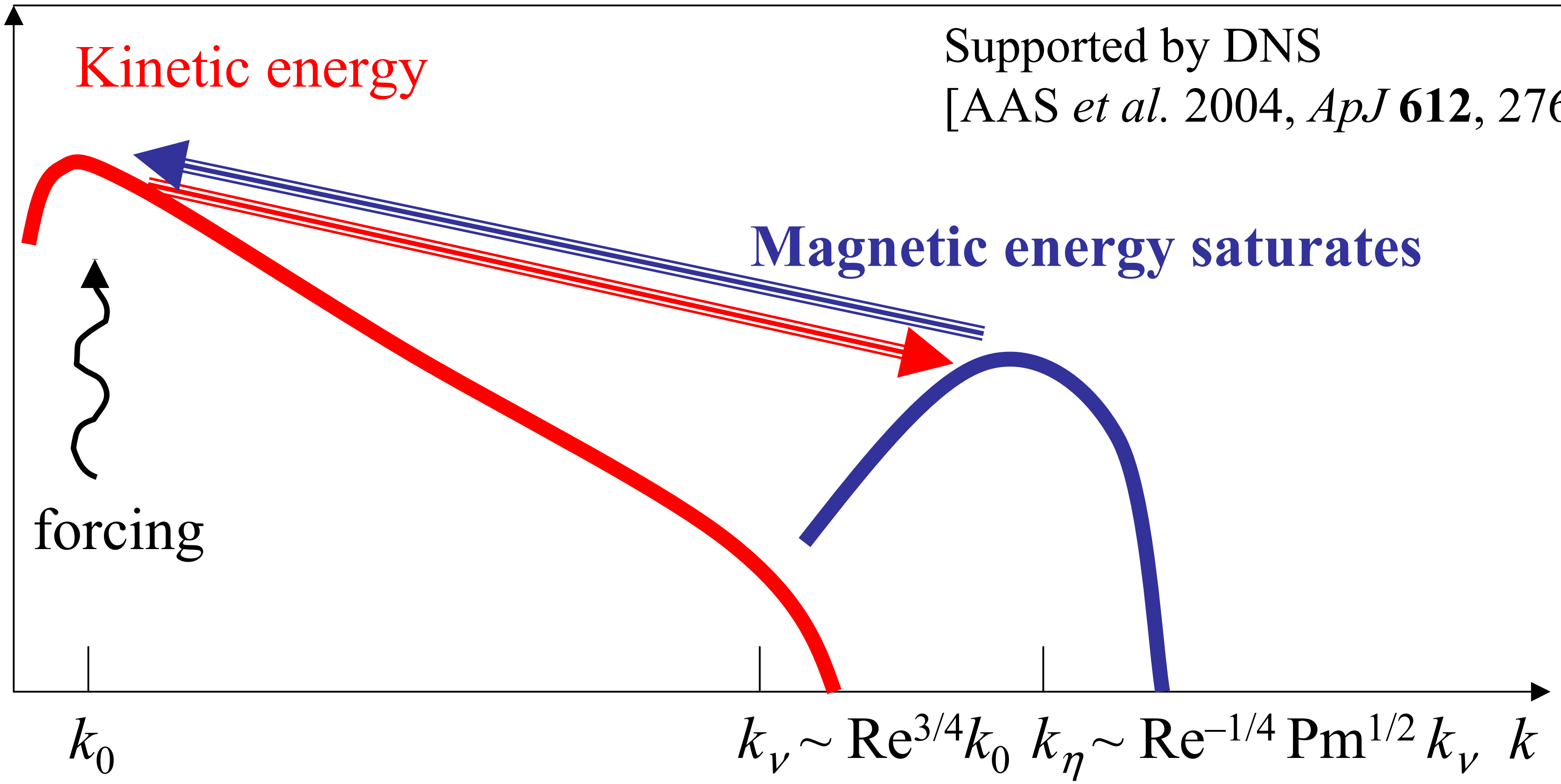
Saturation



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[AAS *et al*. 2004, *ApJ* **612**, 276]

Nonlinear growth/selective decay/fold elongation continue until

$$l_\mathrm{s} \sim l_0 \quad \longrightarrow \quad \boxed{\langle B^2\rangle \sim \langle u^2\rangle}$$

and $l_\eta \sim \mathrm{Rm}^{-1/2}\, l_0 \sim \mathrm{Re}^{1/4}\,\mathrm{Pm}^{-1/2}\, l_\nu$

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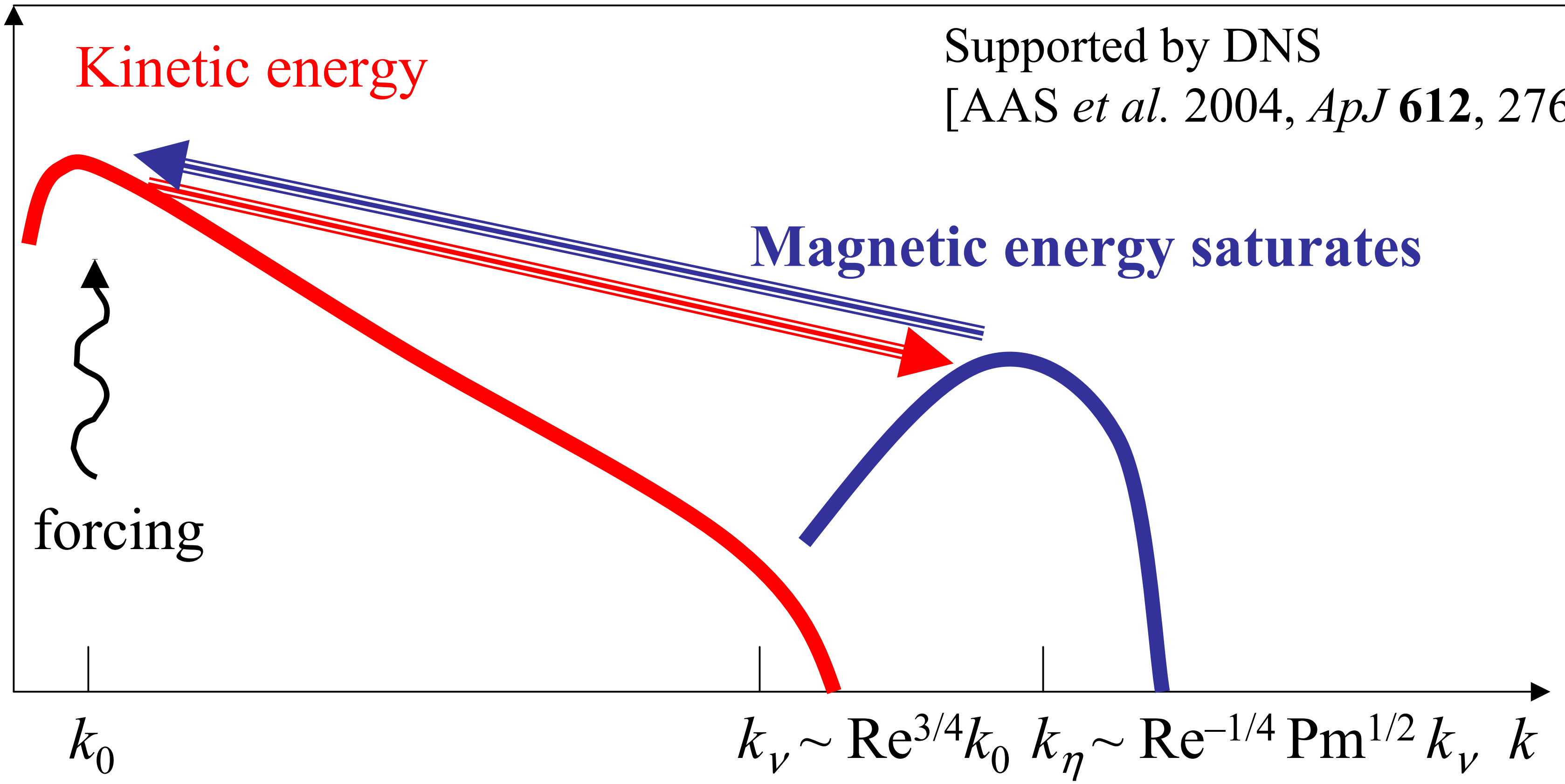

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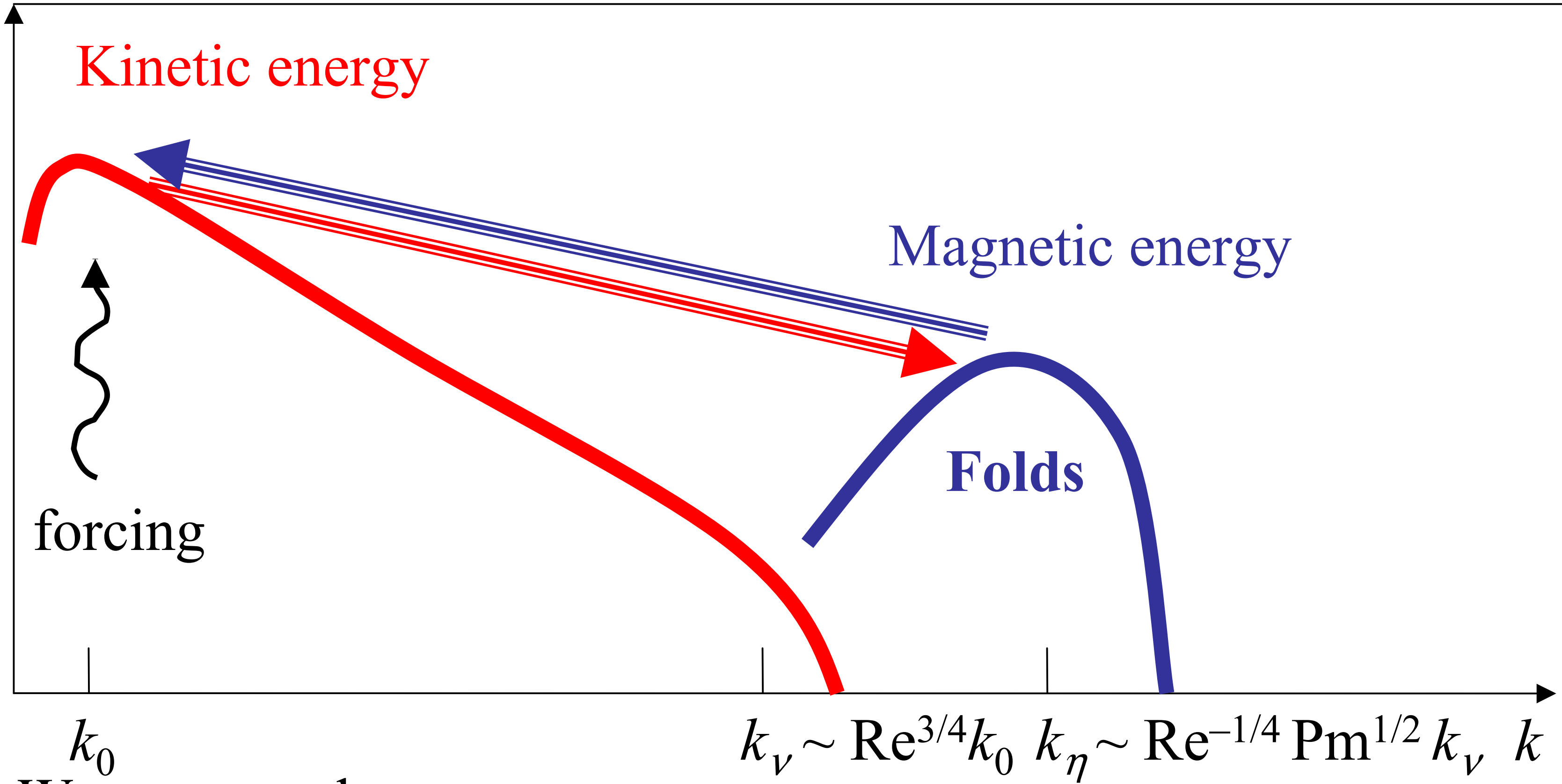
NB: l_η and l_ν distinguishable only if **Pm >> Re$^{1/2}$ >> 1**!!!

Saturation

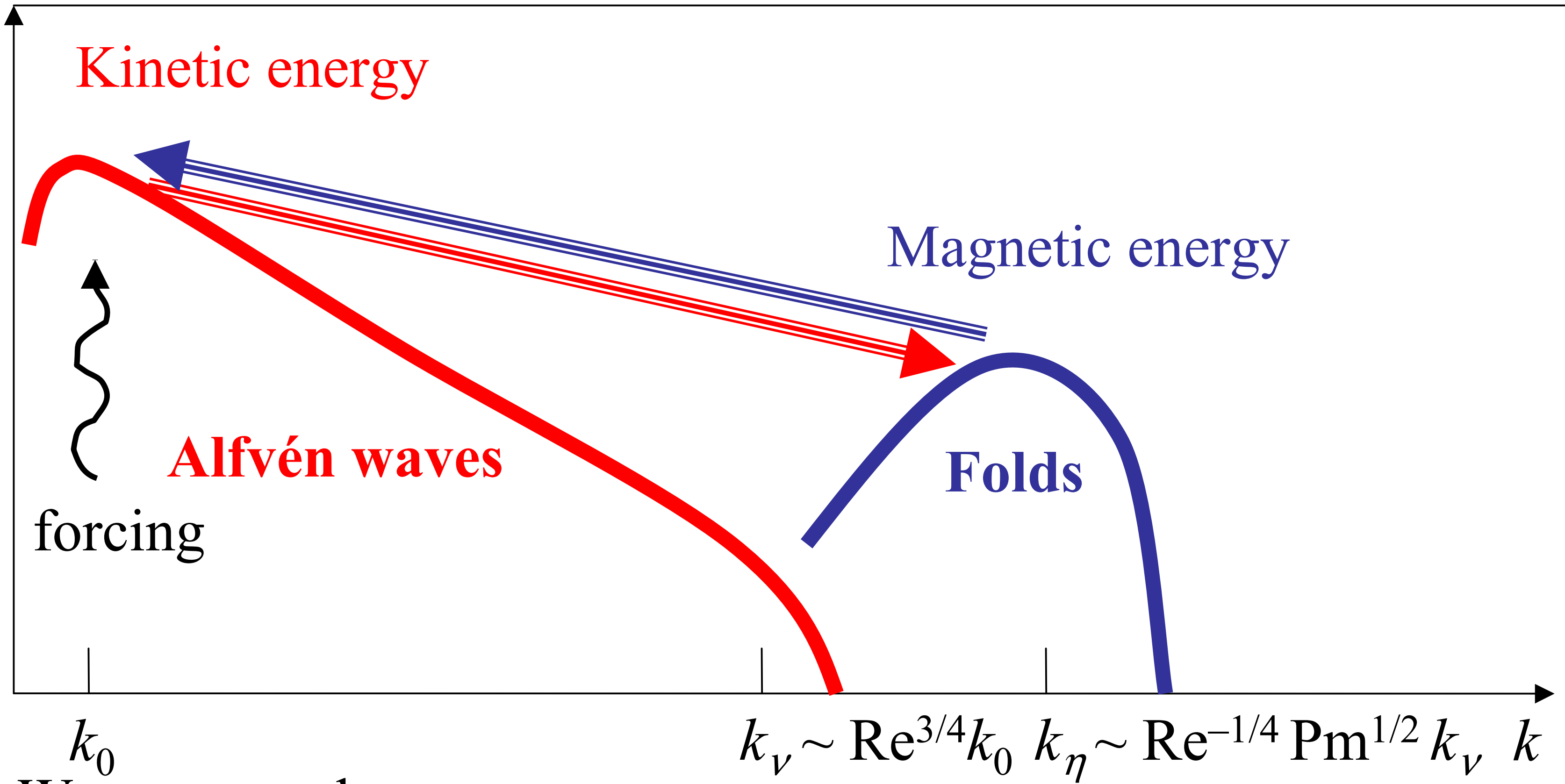


We propose that

- ***saturation*** *is a balance between stretching and mixing by the outer-scale motions and Ohmic diffusion of the folded field*

[Maron *et al.* 2004, *ApJ* **603**, 569; AAS *et al.* 2004, *ApJ* **612**, 276]

Alfvén Waves and Folded Fields



We propose that

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- *the fully developed isotropic MHD turbulence **in the inertial range** is a superposition of folded magnetic fields and Alfvén waves*

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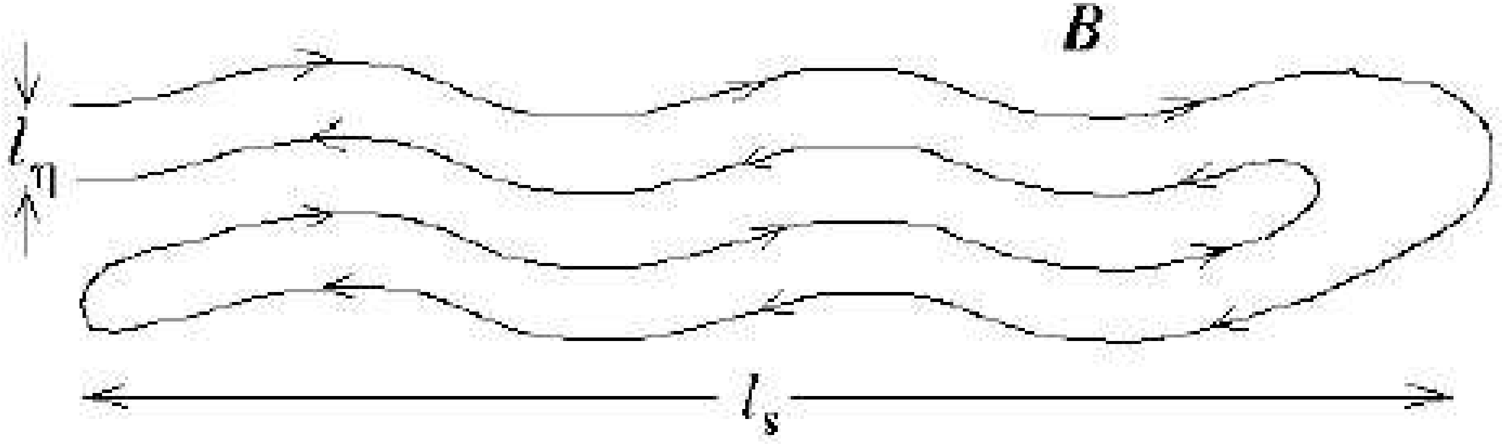

Dispersion relation: $\boxed{\omega^2 = \mathbf{kk} : \hat{\mathbf{b}}\hat{\mathbf{b}}\langle B^2\rangle}$ for $k_\nu \ll k \ll k_\| \sim \ell_s^{-1}$

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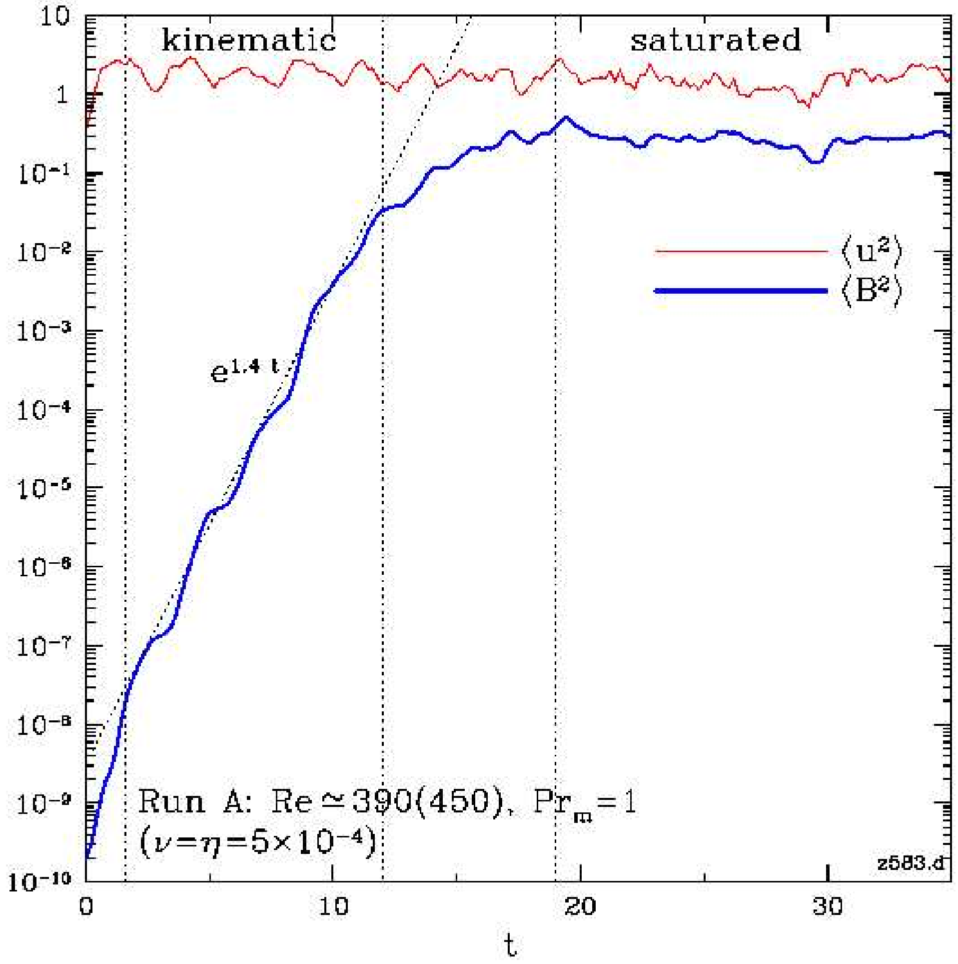
DNS: Intermediate Growth



Slower than exponential growth

[AAS *et al.* 2004, *ApJ* **612**, 276]

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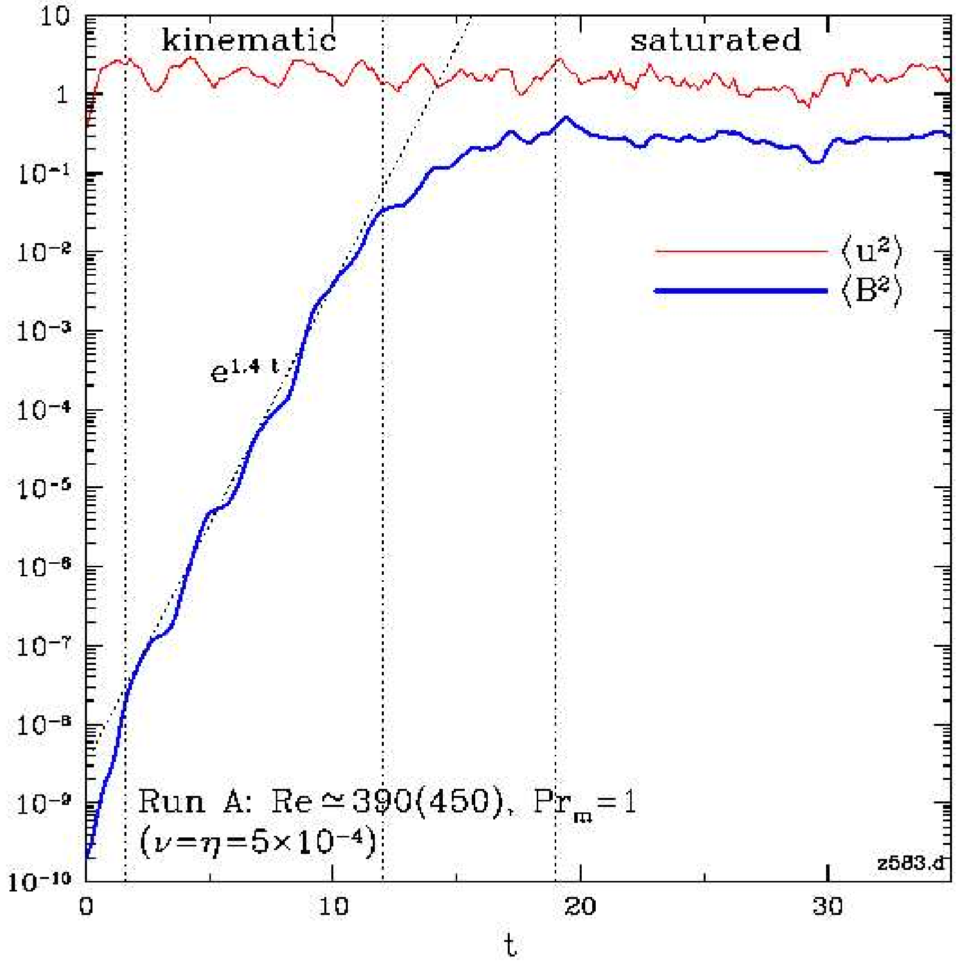

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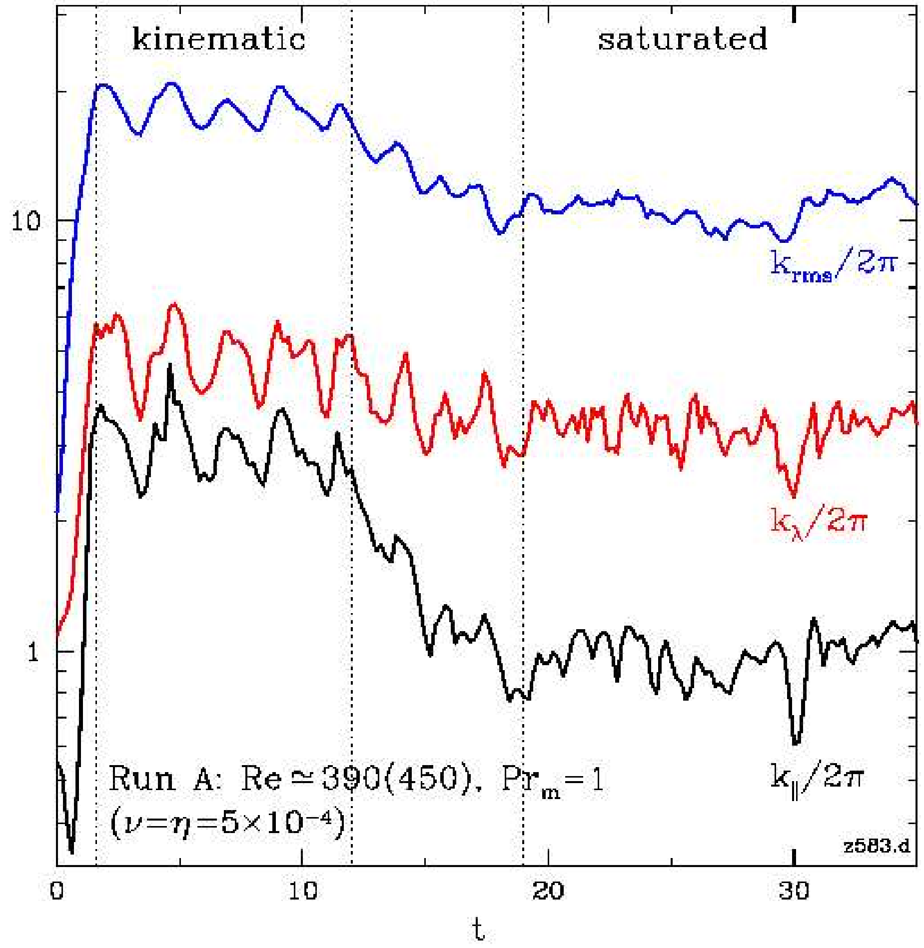

Selective decay and fold elongation

[AAS *et al.* 2004, *ApJ* **612**, 276]

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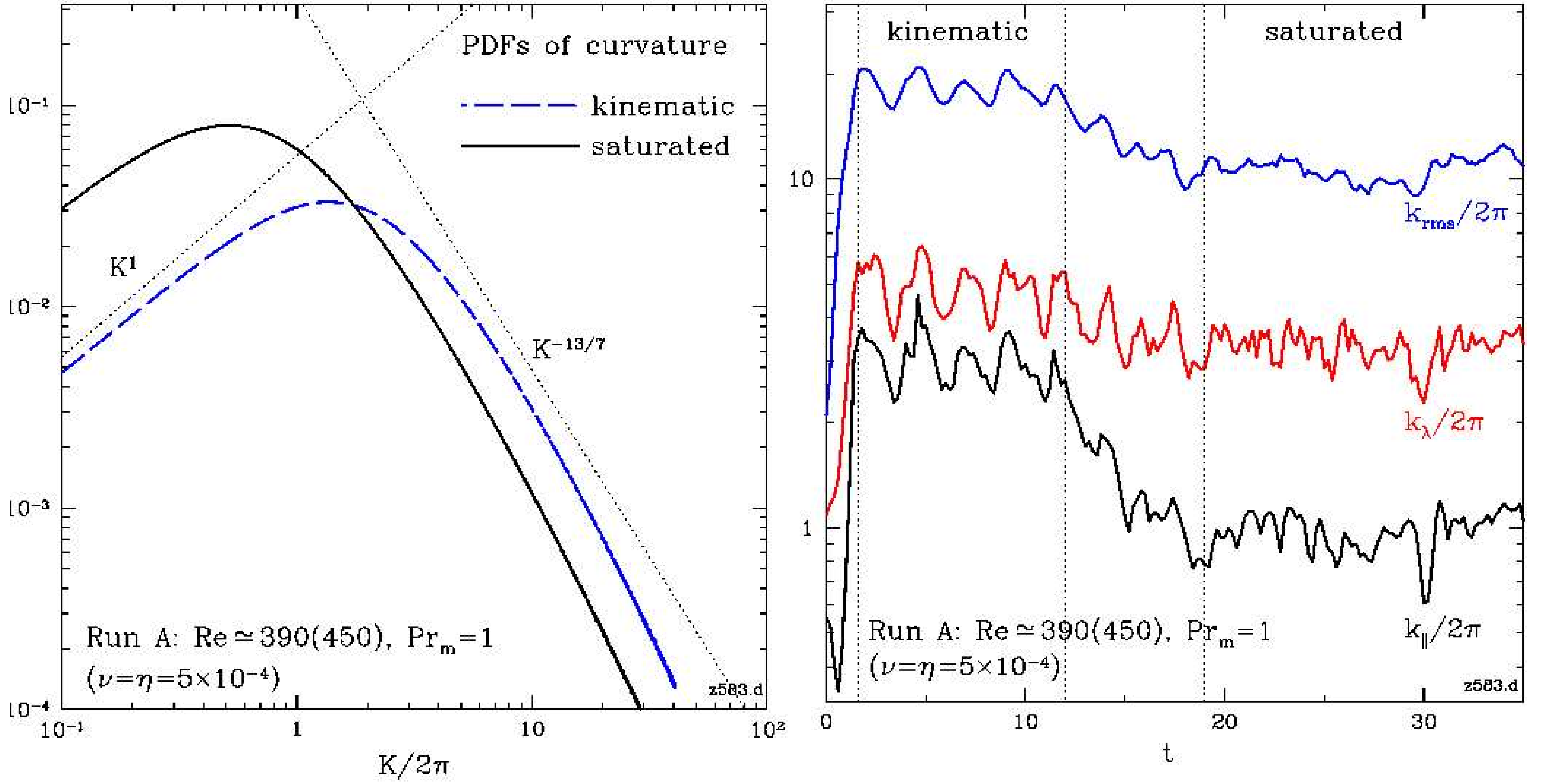

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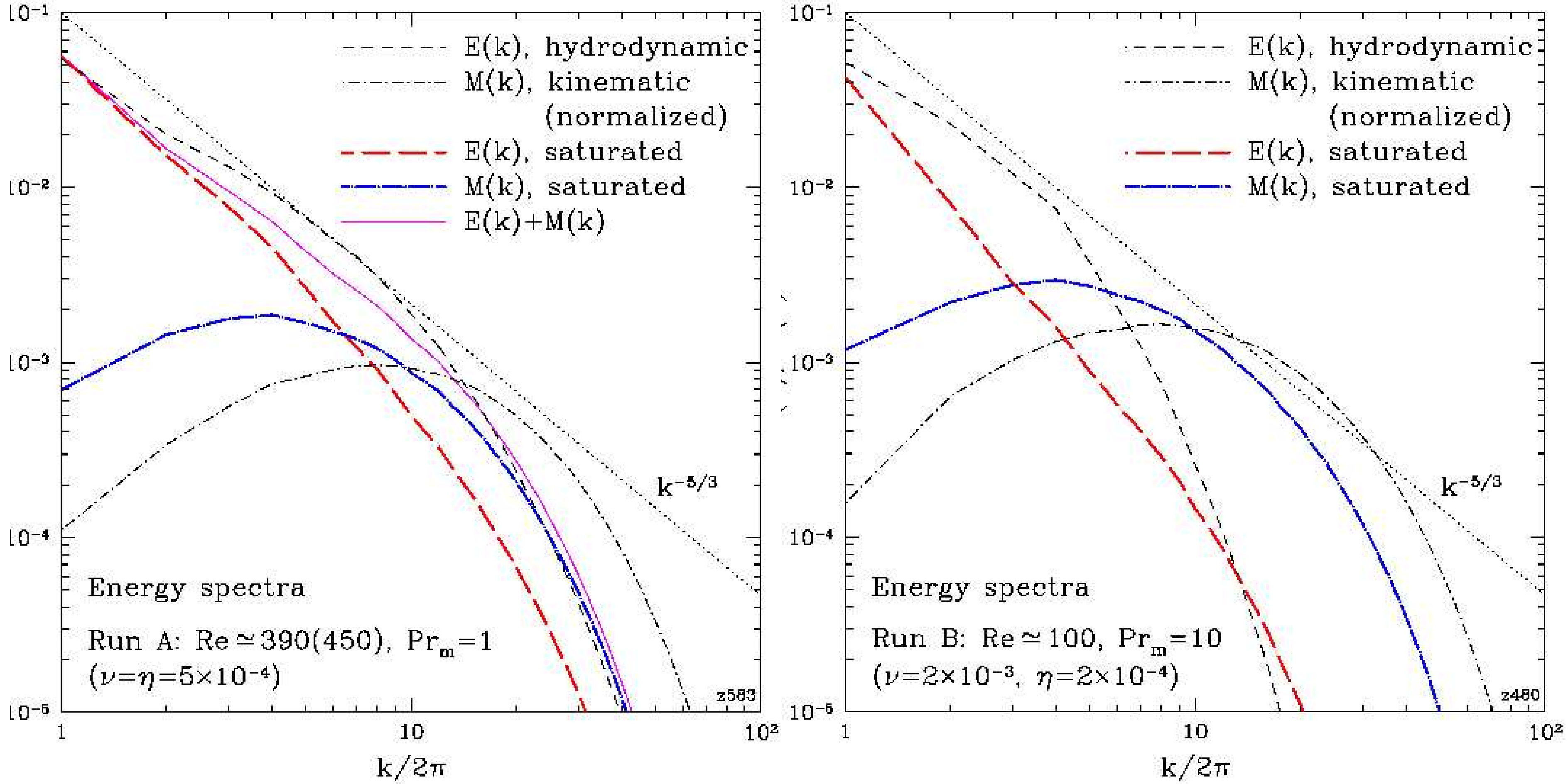
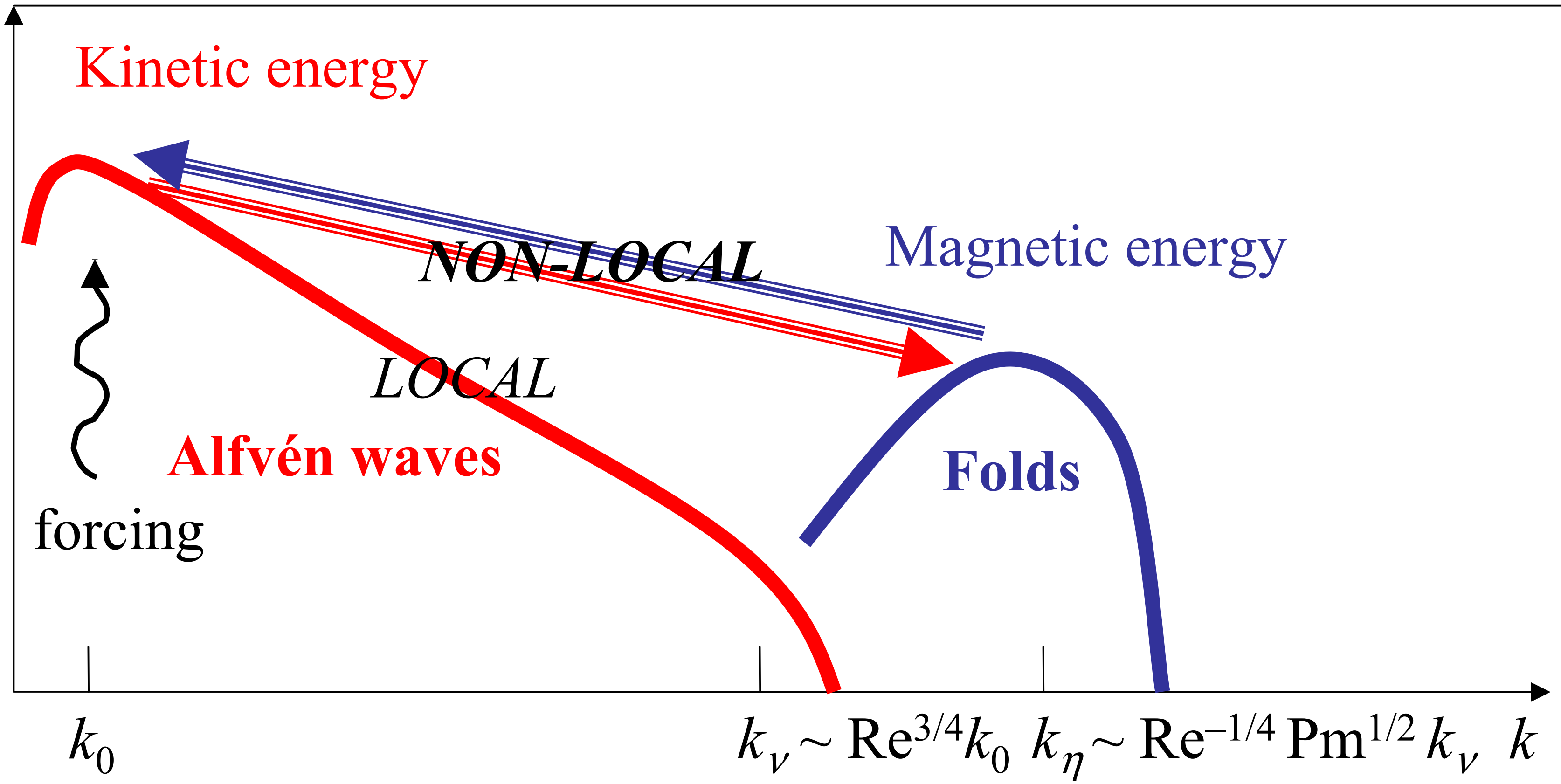
DNS: Saturated Spectra



- **Folds** account for the predominance of large-*k* modes in magnetic-energy spectra
- **Alfvén waves** should show up *in the velocity spectra*

[AAS *et al.* 2004, *ApJ* **612**, 276]

Isotropic MHD Turbulence



Interactions are NOT local in k space!

[AAS *et al.* 2004, *ApJ* **612**, 276]

Isotropic MHD Turbulence

$$\boxed{\begin{gathered}\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = \nu\Delta\mathbf{u} - \nabla p + \mathbf{B}\cdot\nabla\mathbf{B} + \mathbf{f}, \quad \nabla\cdot\mathbf{u} = 0 \\ \partial_t \mathbf{B} + \mathbf{u}\cdot\nabla\mathbf{B} = \mathbf{B}\cdot\nabla\mathbf{u} + \eta\Delta\mathbf{B}\end{gathered}}$$

• **Is there universality?**	**?**
• **Are interactions local?**	**NO**
• **Are macro scales important?**	**YES**
• **Are micro scales important?**	**YES**
• **Are magnetic and velocity fluctuations in scale-by-scale equilibrium with each other?**	**NO**